Definiteness in Early Set Theory

Laura Crosilla and Øystein Linnebo

Abstract: The notion of definiteness has played a fundamental role in the early developments of set theory. We consider its role in work of Cantor, Zermelo and Weyl. We distinguish two very different forms of definiteness. First, a condition can be definite in the sense that, given any object, either the condition applies to that object or it does not. We call this *intensional definiteness*. Second, a condition or collection can be definite in the sense that, loosely speaking, a totality of its instances or members has been circumscribed. We call this *extensional definiteness*. Whereas intensional definiteness concerns whether an intension applies to objects *considered one by one*, extensional definiteness concerns *the totality of objects* to which the intension applies. We discuss how these two forms of definiteness admit of precise mathematical analyses. We argue that two main types of explication of extensional definiteness are available. One is in terms of completability and coexistence (Cantor), the other is based on a novel idea due to Hermann Weyl and can be roughly expressed in terms of proper demarcation. We submit that the two notions of extensional definiteness that emerges from our investigation enable us to identify and understand some of the most important fault lines in the philosophy and foundations of mathematics.

Keywords: Intensional definiteness, Extensional definiteness, Concept of set, Potentialism, Cantor, Zermelo, Weyl

1. Introduction

Philosophers of mathematics sometimes talk about definiteness and kindred notions. For example, Michael Dummett and others have claimed that the concepts of ordinal number and set are indefinitely extensible, in the sense, roughly, that any definite totality of instances of the concept can be used to define yet another instance, outside of the mentioned totality.[1] It follows that there can be no definite totality of absolutely all ordinals or sets. Others, including both philosophers and mathematicians, react with impatience and incredulity. Furthermore, mainstream mathematicians are today less prone to talk about definiteness.[2] So, one might suspect, the philosophers' notions of definiteness are just detached philosophy and mathematically sterile.[3]

We contend that this reaction is unwarranted. Far from being mathematically sterile, we show that there are philosophically interesting notions of definiteness that lend themselves to precise mathematical explication and ultimately also to some interesting mathematics. In particular, during and after the Cantorian revolution that ushered in modern set theory, various notions of definiteness figured in the work of some of the most prominent mathematicians. Debates about these notions helped shape the emerging theory of sets.

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[1] See, e.g., (Dummett, 1963, pp. 195–96), (Dummett, 1991, pp. 316–17), and (Dummett, 1993, p. 441), as well as (Shapiro and Wright, 2006) and (Linnebo, 2018) for discussion and analysis.
[2] As we will see in Section 4, Solomon Feferman is an important exception.
[3] See, e.g., the dismissive attitude found in (Boolos, 1998, p. 224) and (Burgess, 2004, p. 205).

To defend our contention, we discuss some prominent mathematicians' appeals to definiteness in the first half century of Cantorian set theory, roughly between the 1880s and the 1930s. We find it necessary to disentangle two very different forms of definiteness. First, a condition (by which we mean an open formula, possibly with parameters) can be definite in the sense that, given any object, either the condition applies to that object or it does not. We call this *intensional definiteness*. Second, a condition or collection can be definite in the sense that, loosely speaking, a totality of its instances or members has been circumscribed. We call this *extensional definiteness*. Whereas intensional definiteness concerns whether an intension applies to objects *considered one by one*, extensional definiteness concerns *the totality of objects* to which the intension applies. Thus, there are two importantly different families of notions of definiteness. We show that notions from both families are invoked repeatedly in the literature during the emergence of set theory.

We also investigate how these two forms of definiteness admit of precise mathematical analyses. A natural explication of intensional definiteness is in terms of bivalence. That is, a condition $\varphi(x)$ is intensionally definite just in case, for any given object a, either $\varphi(a)$ is true or it is false. For example, 'x is transitive' is intensionally definite on the domain of sets, whereas 'x is small' is not. Of course, since we talk about truth and falsity, this explication takes place in a metalanguage. In the context of intuitionistic logic, a closely related explication is available in the object language, namely, that the Law of Excluded Middle (LEM) holds for the condition:

$$\forall x(\varphi(x) \vee \neg\varphi(x))$$

The idea of extensional definiteness leaves more choice. We distinguish two main types of explication. One takes its departure from the Aristotelian idea of potential infinity. According to Aristotle, only finitely many numbers can coexist; the rest are merely potential. As we will see, Cantor liberalized this idea massively so as to allow various huge infinite collections to coexist, while denying that all sets coexist. To be extensionally definite can then be analyzed as (possible) coexistence. Another explication is based on a novel idea due to the great mathematician Hermann Weyl.[4] Loosely speaking, the idea is that extensional definiteness is a matter of proper demarcation. Remarkably, this intuitive idea is given a precise logical explication. A collection is extensionally definite just in case quantification over that collection "behaves classically". As we shall see, the resulting notion is less demanding than the liberalized Aristotelian notion of possible coexistence.

2. Cantor

Recall that a condition is said to be intensionally definite when, for any given object, either the condition applies to the object or it does not. We begin with some appeals to intensional definiteness in the work of Cantor.

> I call a manifold (an aggregate [Inbegriff], a set) of elements, which belong to any conceptual sphere, *well-defined* [wohldefiniert], if on the basis of its definition and in consequence of the logical principle of excluded middle, it must be recognized that it is internally determined whether an arbitrary object of this conceptual sphere belongs to the manifold or not, and also, whether two objects in the set, in spite of formal differences in the manner in which they are given, are equal or not. (Cantor, 1882)[5]

We see that Cantor characterizes a "manifold" as "well-defined" just in case two conditions are met. ('Manifold' is his generic term for any kind of collection.) First, it must be "determined whether an arbitrary object" of the appropriate sort "belongs to the manifold or not". Second, it must be determined whether any two members of the manifold are equal or not. In short, a "manifold" is well-defined just in case it is associated with an intensionally definite membership condition and a criterion of identity for its members.[6]

A similar characterization of a well-defined collection can be found in work by Dedekind dated 1872–74:

4 See our (Crosilla and Linnebo, 2024).

5 Translation by (Tait, 2009, p. 271).

6 As the quote shows, Cantor also presupposes that the "manifold" belongs to a single "conceptual sphere". He does not, however, explicitly lay that down as a requirement for the manifold to be well-defined, let alone explain how such a requirement is to be understood.

> A *set* or *collection* is determined when for every thing it can be judged whether it belongs to the set or not. (Dedekind 1872, quoted in Ferreirós, 2023, p. 261)

Yet another example from the same period is Frege, whose preferred notion of a collection is that of an extension of a sharply defined concept—or, as we may put it, an intensionally definite concept.[7]

Thus, following Dedekind and Frege, Cantor began with a notion of set that emphasized the provision of an intensionally definite membership condition. This may be slightly more guarded than the so-called naïve conception of sets, which allows any condition to define a set. After all, the three mentioned mathematicians emphasize that the proposed membership condition has to be intensionally definite. Cantor may also assume that the set is contained in a single "conceptual sphere" (see note 6). Regardless, Cantor soon realized that greater care is needed.

An important step in that direction is Cantor's (1895/97, p. 481) famous definition of set:

> By a 'set' we understand every collection into a whole [Zusammenfassung] M of determinate, well-distinguished objects [bestimmten wohlunterschiedenen Objekten] m of our intuition or our thought (which will be called the 'elements' of M). We write this as: $M = \{m\}$. (Cantor 1895, p. 481; translation by Florio and Linnebo, 2021).

Here we find a somewhat different characterization of what a set is. We start with some objects m. Indeed, since we may start with a plurality of objects, it is tempting to use the resources of contemporary plural logic and say that we start with one or more objects mm.[8] These objects are then "collected" or gathered into a whole M, which is the set of the objects in question. Using plural logic, we may write this as $M = \{mm\}$. But again, successful definition of a set is not guaranteed. Cantor requires the operation of collecting many objects into a single set to be applied to some "determinate, well-distinguished objects" mm.

How should this requirement be understood? At the very least, the notion of being one of the objects mm to be collected into a single set must be intensionally definite. That is, for any given object x it must be determined whether x is one of mm or not. But the requirement that the operation of collecting into a whole be applied to some "determinate, well-distinguished" objects can also be understood to demand more. To see how, a distinction is useful. A property of some objects is said to be *distributive* just in case the ascription of the property to some objects is equivalent to the ascription of it to each of these objects. By contrast, a property is said to be *collective* or *non-distributive* when it says something about all of the objects together. For example, 'the students are French' involves a distributive predicate, whereas 'the students surround the building' involves a collective one. Returning to the property of being some "determinate, well-distinguished objects", the question is whether this should be understood in a distributive or a collective way.

We submit that the collective reading is more plausible. To have some objects that can be collected into a single set it is not enough that each of the objects is "determinate" and that any two of them are "well-distinguished"; rather, the objects in question have to be *collectively* determinate, that is, we must have pinned down which objects *they* are. As we will see shortly, this is also the reading that Cantor later says he intended. It is plausible, therefore, to understand Cantor's 1895 characterization of a set as requiring that a set be *extensionally* definite.

In fact, even prior to 1895 Cantor appears to have invoked the idea of extensional definiteness. One example concerns the generative approach of (Cantor, 1883) to transfinite numbers, which are close to what we now call ordinal numbers.[9] His first principle of number generation states that for any number α, we may add one so as to obtain its successor $\alpha + 1$. A more interesting second principle states that:

> if any definite [bestimmte] succession of defined integers is put forward of which no greatest exists, a new number is created by means of this second principle of generation, which is thought of as the *limit* of those numbers; that is, it is defined as the next number greater than all of them. (Cantor, 1883, pp. 907–908)

7 See esp. (Frege, 1893), as well as (Parsons, 2012) for discussion.
8 See (Boolos, 1984) for a seminal contribution, as well as (Florio and Linnebo, 2021) for a recent survey and assessment.
9 See (Tait, 2009, pp. 281-82) for discussion.

In short, the second principle allows us to form the least upper bound of any "definite succession" of numbers.[10]

Assume, following Cantor, that the natural numbers form a "definite succession". Then the second principle of number generation allows us to define their limit, namely ω. Repeated application of the first principle now yields all numbers of the form $\omega + n$, for n a natural number. Since this too is a "definite succession", we can take their limit, so as to obtain $\omega + \omega$. Clearly, we can generate larger and larger ordinal numbers. This yields what Cantor calls "the extended number sequence" (p. 912).

What is it for some numbers to be a "definite succession" and thus, by the second generating principle, to possess a limit or least upper bound? We submit that this is best understood as some form of extensional definiteness. Unlike the sequence of natural numbers, which is surpassed by their limit, ω, the extended number sequence is "absolutely infinite" (p. 916). As such, it has no limit but (says Cantor, quoting Albrecht von Haller) "lie[s] always ahead of me" (ibid.). This fundamental difference between the natural number sequence and the extended number sequence cannot be explained in terms of intensional definiteness alone. For applied to any given object, the notion of being an extended number is just as clear and well-defined as that of being a natural number: either this object is an (extended) number or it is not. The difference pertains rather to the extended number sequence *in its entirety*, namely, that it is not extensionally definite. Thus, we contend, Cantor's notion of a "definite succession" is best understood as involving some form of *extensional* definiteness.[11]

A second example of extensional definiteness in Cantor prior to 1895 is found in his critical 1885 review of (Frege, 1953). Cantor complains that Frege "overlooks altogether the fact that the 'extension of a concept' is, in general, something quantitatively completely indeterminate" (Ebert and Rossberg, 2009, p. 346).[12] Whether the extension of a concept is "quantitatively determinate" is, we contend, a question about extensional definiteness, not intensional. First, Cantor makes it explicit that the question concerns the *extension* of a concept, not the concept itself. Second, the word "quantitatively" makes it clear that the question is whether all the instances of the concept, *taken together*, can be assigned a number or quantity, not whether it is well-defined what it is for the concept to apply to any given *single* object.

Let us return to our discussion of what it takes for a set to be well-defined. We argued that it is most plausible to interpret Cantor's famous 1895 characterization of a set as requiring that the one or more objects to be collected together into a whole (namely, a set) be collectively well-defined or circumscribed. That is, we find it natural to read Cantor 1895 as requiring that a set be extensionally, not just intensionally, definite. After the set-theoretic paradoxes were discovered and started to be discussed among German mathematicians around 1897, Cantor wrote two letters that clearly and unequivocally make this requirement. These letters also take some important steps towards explaining how he understood this notion of extensional definiteness.

We begin with an 1897 letter to Hilbert.

> I say of a set that it can be thought of as *finished* [...] if it is possible without contradiction (as can be done with finite sets) to think of *all its elements as existing together*, and so to think of the set itself as *a compounded thing for itself*; or (in other words) if it is *possible* to imagine the set as *actually existing* with the totality of its elements. (Ewald, 1996, p. 927)

Cantor proceeds to commend the French and Italian words for set ('ensemble' and 'insieme', respectively), which emphasize that the elements of a set must exist together. He continues:

> And so too in the first article of [Cantor 1895, which was quoted above], I define 'set' (meaning thereby only the finite or transfinite) at the very beginning as an 'assembling together' [Zusammenfassung]. But an 'assembling together' is only possible if an '*existing together*' [*Zusammensein*] is *possible*. (928)

10 Cantor's use of the second principle makes it clear that 'integer', in the quoted passage, is used as synonymous with 'number'. A third principle is also formulated, which need not concern us here.

11 Cantor thus comes close to the Burali-Forti paradox, which is the paradox of the ordinal associated with the well-ordered sequence of all ordinal numbers. There is a scholarly debate about whether or not Cantor already in 1883 has some understanding of this paradox; cf. (Moore and Garciadiego, 1981), (Ferreirós, 1999, p. 292) and (Tait, 2009, p. 281). We need not here take a stand on this question.

12 There is a scholarly debate about whether this should be seen as warning Frege of the disaster that struck seventeen years later when he received a letter from Bertrand Russell (Tait, 2009; Ebert and Rossberg, 2009). Cf. also (Dummett, 1994, p. 26]) who argued that Cantor was well ahead of Frege in seeing the importance of indefinite extensibility. Once again, we need not take a stand on this matter.

We wish to make two observations. First, the property of "existing together" is clearly collective, not distributive. So Cantor in 1897 reads his own 1895 characterization of a set as requiring that a set be extensionally, not just intensionally, definite. Second, the intended notion of extensional definiteness is now becoming clearer. It is a matter of all of the elements of the desired set coexisting so that the set resulting from collecting them together can be regarded as "finished".

An 1899 letter to Dedekind continues this explication of the sense in which each set is extensionally definite.

> [I]t is necessary [...] to distinguish two kinds of multiplicities (by this I always mean *definite* multiplicities).
>
> For a multiplicity can be such that the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*. [...]
>
> If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into '*one* thing', I call it a *consistent multiplicity* or a 'set'. (Ewald, 1996, p. 931-932)

Cantor's thought here appears to be that there is an intrinsic difference between multiplicities that form sets and multiplicities that do not, and that this intrinsic difference *explains* why some but not all multiplicities are eligible for set formation. A set is characterized as a "finished" collection, all of whose elements can "exist together" or be imagined as "actually existing". So for a multiplicity to be eligible for set formation, it must be capable of being regarded as "finished", and its elements must be capable of "existing together". A multiplicity that is capable of the sort of "completion" is thus intrinsically suited for set formation, whereas a multiplicity that resist such "completion" is intrinsically unsuited for set formation. For instance, since the multiplicities of everything thinkable and of all ordinals resist "completion", there can be no universal set or set of all ordinals.

Some will complain that the quasi-temporal language of coexistence and being finished has no place in the analysis of an abstract subject matter that exists independently of us. It must be admitted that Cantor's remarks about legitimate set formation [Zusammenfassung] and his associated response to the paradoxes are loose and underdeveloped. We nonetheless find his remarks suggestive.

We would therefore like to propose a way to understand Cantor's remarks about coexistence and being finished on which these remarks have genuine explanatory potential.[13] The idea is to understand Cantor's notion of extensional definiteness as a potentialist notion of *completability*. Let us explain. In stark contrast to Cantor, Aristotle held that only potential infinities are coherent, not actual ones. For example, however many natural number are instantiated—and thus, according to Aristotle, exist—it is possible to instantiate yet more. Writing '$\textsc{Succ}(m,n)$' for the claim that m is directly succeeded by n, he thus endorses:

$$\Box\forall m\Diamond\exists n\,\textsc{Succ}(m,n) \tag{1}$$

However, Aristotle denies that it is possible to complete this process of producing successors of natural numbers:

$$\neg\Diamond\forall m\exists n\,\textsc{Succ}(m,n) \tag{2}$$

Another way to put this is that the notion $\mathbb{N}(x)$ of being a natural number is incompletable, in the sense that it is impossible for some objects to coexist which are all the numbers there ever could be:

$$\neg\Diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow \mathbb{N}(y)) \tag{3}$$

We here treat every plurality as "finished". That is, when we generalize plurally over one or more objects, we assume that the objects in question coexist.[14]

[13] See (Parsons, 1983b), (Linnebo, 2013) and (Studd, 2013) for attempts to realize that explanatory potential.

[14] These are fairly standard assumptions in the contemporary literature on the modal logic of plurals; see (Florio and Linnebo, 2021, ch. 10 and references therein). The contemporary notion of plurality differs, in this regard, from Cantor's less demanding notion of a "multiplicity", which permits multiplicities to be "inconsistent' or "unfinished". As Charles Parsons (1977, Sect. III) observes, Cantor's inconsistent multiplicities are better understood as Fregean concepts than some given objects.

Cantor, of course, is famous for rejecting Aristotle's claims (2) and (3). The natural numbers are completable, he argues, and therefore form a set. Nevertheless, in the quoted letters Cantor can be understood as making claims that are logically analogous to Aristotle's. Let us write '$\textsc{Set}(xx,y)$' for "y is the set obtained by collecting xx into a whole". Cantor can be understood as claiming that the operation of set formation can be applied to any coexistent objects but that it is impossible to complete "the process" of applying the operation:

$$\Box\forall xx\Diamond\exists y\,\textsc{Set}(xx,y) \tag{4}$$

$$\neg\Diamond\forall xx\exists y\,\textsc{Set}(xx,y) \tag{5}$$

These two claims echo (1) and (2), respectively.

More generally, a condition $\varphi(x)$ is completable just in case it is possible for the generation of instances to be "finished", such that all the instances the condition could ever have are available:

$$\Diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow \varphi(y))$$

The "multiplicity" associated with a condition is thus intrinsically suitable to form a set just in case the condition is completable.

Our proposal is that Cantor's use of temporal and modal language need not be dismissed as just colorful language but can be understood as conveying a valuable idea about set formation. Whenever a mathematical operation is iterated, the metaphor of a process is natural. But the operation of set formation has a special feature, namely, that it takes *many objects* as input and outputs their set. This raises the question of what kinds of input this operation might take. Cantor's attempt at an answer is that availability to serve as input to set formation is a matter of coexistence or being finished. Using the metaphor of a process, he thus identifies a new theoretical primitive of *joint availability* for set formation. This new primitive can be explicated using modality and plural logic; and thus explicated, its explanatory potential can be realized (cf. footnote 13).

Summing up, we have seen that Cantor began with a conception of set that emphasizes the intensional definiteness of every well-defined set but gradually shifted to a conception that adds the requirement that a set be extensionally definite. We have also argued that his mature notion of extensional definiteness can be understood as the potentialist one of all the elements of a set "coexisting", or being jointly available for set formation. While this is far from a worked-out account of legitimate set formation, it at least points the direction for such an account.

3. Zermelo

The notion of definiteness plays a key role in Zermelo's foundational work. It features prominently in his well-known axiomatisation of set theory of 1908 and motivates his subsequent foundational reflections in the 1920s and 1930s. A constant reference of Zermelo is Cantor's set theory. Zermelo repeatedly states that Cantor's "original definition" of set in (Cantor, 1895) had been long recognized as insufficient in view of the antinomies of set theory.[15] Throughout his work Zermelo puts forth a number of original proposals that improve Cantor's definition of set.[16] Here we focus on (Zermelo, 1908b) and some ideas presented in (Zermelo, 1929a) and (Zermelo, 1930c).

At the beginning of (Zermelo, 1908b) Zermelo writes:

> [...] the very existence of [set theory] seems to be threatened by certain contradictions, or 'antinomies', that can be derived from its principles [...] and to which no entirely satisfactory solution has yet been found. (Zermelo, 1908b, p. 189)

15 See e.g. (Zermelo, 1908b, p. 189-91) and (Zermelo, 1929b, p. 38). Moore (1978) argues that Zermelo's axiomatisation was primarily motivated not by the paradoxes but by the desire to reply to widespread criticism of his 1904 well-ordering theorem.

16 Zermelo's ideas are difficult to interpret as they lack a precise framework and language. We think, however, that Zermelo offers a significant and original perspective on definiteness and for this reason we review its key characteristics. For detailed analysis of Zermelo's thought on definiteness see, for example, (Ebbinghaus, 2003), (Ebbinghaus, 2010) and (Felgner, 2010).

He claims that in view of "Russell's antinomy" it is no longer admissible "to assign to an arbitrary logically definable notion a 'set', or 'class', as its 'extension' ". This, Zermelo argues, shows that

> Cantor's original [viz. 1895] definition of a 'set' as 'a collection, gathered into a whole, of certain well-distinguished objects of our perception or of our thought' therefore certainly requires some restrictions [...]. (Zermelo, 1908b, p. 189-91)

To overcome these difficulties, Zermelo presents an axiom system intended to be sufficiently restricted to exclude all contradictions, but also sufficiently wide to retain all that is valuable in set theory. Zermelo's key restrictions are introduced in his *schema of separation* (Axiom III) and it is here that definiteness comes into play. Zermelo's separation allows us to define a *subset of a given set*, say M, by separating out of M those of its elements that satisfy a "definite" property.

In (Zermelo, 1908b, p. 195) the separation schema reads as follows:[17]

> Whenever the propositional function $\mathfrak{E}(x)$ is definite for all elements of a set M, M possesses a subset $M_{\mathfrak{E}}$ containing as elements precisely those elements x of M for which $\mathfrak{E}(x)$ is true.

Separation is therefore more restrictive than naive comprehension in two respects:

- it defines a subset of *an already given set*, M, by separating out those elements of M that satisfy some property;
- it requires that the property used to define the subset of M is *definite*.

The first restriction is fundamental, as it aims at excluding "contradictory notions such as 'the set of all sets' or 'the set of all ordinal numbers' " (Zermelo, 1908b, p. 195). The second restriction is needed to avoid paradoxes such as Richard's paradox. The idea is that notions such as 'definable by means of a finite number of words' are not definite, so that separation cannot be applied to them. This second restriction is also highly significant, since Zermelo's axiomatisation is not expressed on the basis of a fixed formal language. The restriction is intended to rule out problematic notions such as the notion of definability that figures in Richard paradox. As Zermelo retrospectively observed, "a universally acknowledged 'mathematical logic' on which I could have relied did not exist" at the time (Zermelo, 1929b, p. 359). Therefore in 1908 he appeals to definitenss to excise problematic properties. However, lacking a rigorous description of the language of set theory, Zermelo's notion of definiteness was soon criticised for being too vague. Before looking at this criticism, we need to examine in more detail how Zermelo characterises definiteness and how his notion relates to our distinction between intensional and extensional definiteness.

Zermelo (1908, p. 191) starts by presupposing a given domain, B, and a system of *fundamental relations* over B. He then offers the following explanation of definiteness (Zermelo, 1908, p. 193):

> A question or assertion $\mathfrak{E}$ is said to be "*definite*" if the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not. Likewise, a "propositional function" $\mathfrak{E}(x)$ in which the variable term x ranges over all individuals of a class $\mathfrak{K}$, is said to be "*definite*" if it is definite for *each single* individual x of the class $\mathfrak{K}$.

[17] Zermelo's schema of separation figures also in (Zermelo, 1908a), where Zermelo employs the expression "well-defined property" ("wohldefinierte Eigenschaft"). More precisely in that text the separation axiom reads as follows: "All elements of a set M that have a property E well-defined for every single element are the elements of another set, M_E, a 'subset' of M" (Zermelo, 1908a, p. 121). It is clear that definiteness is here to be read distributively, as applying to each individual element of the set M. As argued in (Felgner, 2010, p. 180), at first Zermelo's terminology reminds us of Cantor's 1882 definition of set in terms of "wohldefinierte Mannigfaltigkeiten", which was quoted at page 44. Similar terminology appears in lecture notes by Zermelo from the winter semester 1900/1901 (with additions from the years 1904-1906). Felgner argues that when later Zermelo employed the term "definit", he borrowed it from Husserl. See also (Moore, 1982, pp. 155-6) for a discussion of earlier attempts by Zermelo to formulate separation.

Zermelo's passage suggests that definiteness is determined by three components: (a) the fundamental relations of the domain, (b) the axioms of set theory and (c) the laws of logic. To this we need to add the claim (d) that such a determination ought to be non-arbitrary. But how is this meant to eliminate the threat of paradoxes such as Richard's? Presumably we can reason as follows. Suppose we start from the fundamental relations of set theory. These are of the form $a \in b$ and $a = b$, for a and b elements of the domain B. Given these fundamental relations we do not have a non-arbitrary way of determining, by employing only the laws of logic, whether a property such as 'definable by means of a finite number of words' holds of some number or not. Therefore we cannot apply separation in such cases.

Given this characterisation of definiteness, we may now wonder how it features within our distinction between intensional and extensional definiteness. After stating the axiom of separation, Zermelo writes that the criterion defining a subset of a set M is definite if "for each single element x of M the 'fundamental relations of the domain' must determine whether it holds or not" (Zermelo, 1908, p. 195). This explanation makes it clear that Zermelo understands definiteness *distributively*, as a definite condition either holds or does not hold of *each individual element* of a given set. In this respect, definiteness looks like a form of *intensional definiteness*.

As we have seen, Zermelo presents his definition of set as an improvement of Cantor's original (1895) definition, which, we argued, can be plausibly read as requiring that a set be extensionally definite. As we find plausible to read Zermelo's definiteness as a form of intensional definiteness, one may wonder how this can result in an improvement with respect to Cantor's definition of set. We have two main observations in this respect. First, there is substantial improvement, of course, in Zermelo's *axiomatic* presentation of set theory, which affords a more rigorous account of sets. While Cantor may require that a set be extensionally definite, his definition does not furnish clear criteria for sethood, which is something Zermelo's axiomatization strives to do. It is true that Zermelo's separation schema runs into difficulties as it is not sufficiently precise, as we will see shortly. But Zermelo's approach made a rigorous formulation of separation urgent, prompting further improvements to the formulation of set theory.

Second, in the separation schema we find a distinctive interplay between intensional definiteness and the restriction of the defining condition to an already given set, M. It is plausible that Zermelo's separation schema is meant to fix the extension of (the property expressed by) a definite condition relative to the given set M, while M is thought of as extensionally definite. The idea seems to be that as we single out a subset of an already given set M by means of a definite condition, we also fully determine the extension of that condition when its range of application is restricted to the elements of the set M.

This seems a reasonable reconstruction of how definiteness is supposed to restrict and improve on Cantor's definition of set. Let us return now to the complaint that Zermelo's notion of definiteness is insufficiently precise to achieve its purpose. Zermelo's characterisation of definiteness is imprecise, as it lacks the required detail that would allow us to decide in each case whether a property is definite or not. For example, Zermelo does not specify the language in which the axioms of set theory are formulated. In addition, an explication of definiteness that refers to the axioms of set theory is problematic, since definiteness is required to make sense of the separation axiom. Indeed, the lack of precision and the resulting vagueness of Zermelo's notion of 'definite property' was criticised by mathematicians such as Weyl, Fraenkel and Skolem (Weyl, 1910, 1918; Fraenkel, 1922; Skolem, 1922). They proposed improvements leading to the *first-order* formulation of Zermelo-Fraenkel set theory we are familiar with. We will discuss Weyl's proposal in Section 4.[18]

The reflection on Zermelo's notion of definiteness that followed his axiomatisation of set theory helped shape set theory. The proposed solution to the question of how to define definite properties developed in a fully rigorous way ideas already implicit in Zermelo's informal explication of definiteness, in particular components (a)–(c). The idea was to make precise the language of set theory in terms of the (now) ordinary first-order language, with the equality and membership predicates as basic. The legitimate (definite) properties are those that are expressible by repeated application of the first-order logical operations to atomic statements of the form $a = b$ and $a \in b$. The thought was that this suffices to rule out problematic definitions such as that of a number definable by means of a finite number of words. As a consequence, the explicit reference to definiteness that

[18] See (Ebbinghaus, 2003, 2010; Felgner, 2010; Taylor, 2002) for discussions of Zermelo's contemporaries' criticism of his notion of definiteness and for an analysis of Zermelo's different proposals on how to spell out definiteness.

characterised Zermelo's 1908's formulation of the separation schema disappeared. There was no further need to distinguish the definite properties from those that are not, as the language, once appropriately regimented, ensured that only definite properties are expressible.

It is natural to wonder, though, whether the resulting explication of definiteness in terms of definability in the first-order language of set theory is satisfactory. Both directions can be challenged: we will now look at Zermelo's challenge to the thought that every definite property is definabile in the first-order language of set theory. In the next section we will discuss Weyl's misgivings about the reverse inclusion.

In a paper from 1929 and in other texts from the 1930s, Zermelo explored new ways of clarifying the notion of definiteness, sketching new intriguing ideas.[19] Zermelo's new reflections on definiteness followed a number of distinct threads without reaching a definitive and clear solution to the difficulties that afflicted his original approach to definiteness. In fact, Ebbinghaus (2003) argues that Zermelo developed contrasting ideas, differing for the choice of language and the view of the nature of the set-theoretic universe, but did not succeed in bringing them together into a unified approach (Ebbinghaus, 2003, p. 198).

Of particular significance for our investigation is the text "On the concept of definiteness in axiomatics" (Zermelo, 1929). There Zermelo replies to the criticism levelled against his earlier (Zermelo, 1908) formulation of definiteness, stating that he had not been understood. He then scrutinizes his critics' proposed solutions. Among them is Fraenkel's approach to definiteness, that Zermelo terms "genetic". This may be considered a variant of the approach proposed by Weyl (1910) and Skolem (1922), which replaces definiteness by first-order definability in the language of set theory. One of Zermelo's complaints is that the genetic approach makes essential use of the natural numbers, in the form of the *finite* repetitions of the logical operations. It therefore presupposes the natural numbers instead of grounding them set-theoretically.[20] Another complaint is that a genetic characterization of definiteness "contradicts [the] purpose and nature of the *axiomatic* method" (Zermelo, 1929, p.359). Zermelo proposes instead what he calls an "axiomatic" approach, that is, he gives a series of conditions that are to be satisfied by the collection of definite properties. In this new attempt to pin down the notion of definiteness, Zermelo simultaneously hints at two distinct ideas, one model-theoretic and one syntactic.[21]

The first is presented as follows. Zermelo introduces the notion of *logically closed system*, which is, essentially, the deductive closure of a set of axioms (Zermelo, 1929, p. 361). He claims that if a logically closed system is consistent,

> "then it must be 'realizable' as well, that is, representable by means of a "model", by means of a *complete matrix* of the '*fundamental relations*' that occur in the axioms or in the system".

Now, each fundamental relation R is '*disjunctive*', in the sense that either R or its negation holds in the model. The same holds also for composite relations, so that "it is uniquely decided in every model by means of the matrix of the fundamental relations whether or not they hold in it." The definite properties are then those that are "decided by means of the fundamental relations in every model". Zermelo (1929, p. 361-63) also states that

> "[d]efinite" is thus what is *decided in every* single *model*, but may be decided differently in different models; "decidedness" refers to the individual *model*, whereas "definiteness" itself refers to the *relation* under consideration and to the entire *system*.

This enables Zermelo to claim that non-definite properties are those that are either not uniquely determined by the fundamental relations or "alien to the system". For example, the property "not definable by means of a finite number of words in any European language" is not uniquely determined by the fundamental relations, while the property of being a set "painted in green" is alien to the system. To sum up, the key idea of this model-theoretic

[19] See, for example, (Zermelo, 1929, 1930c, 1930a, 1930b).

[20] See page 52 (below) for a passage by Zermelo in which he considers, but then quickly dismisses, a genetic characterisation of definiteness. The reliance of a genetic characterisation of definiteness on the natural numbers is also an important element of Weyl's reflection.

[21] It should be noted that while Zermelo clearly makes use of second-order quantifiers in his axiomatic characterisation, he does not fully specify the underlying language.

characterisation is that definiteness is what is decided in every model of the axioms of set theory on the basis of the fundamental relations of the domain.

It is tempting to see this as a novel development of components (a) - (d) of the 1908 characterisation of definiteness, now focusing on the models of the axioms of set theory. Zermelo briefly mentions also categorical systems of axioms, but does not explain the relation between his notion of definiteness and categorical systems. Categoricity, indeed, becomes a crucial theme in his foundational investigations in the 1930s, when he tries to develop a notion of set as categorical domain, that is, as a domain that can be characterized by a categorical system of axioms. Perhaps one way to round out Zermelo's 1929 proposal is to suggest that a categorical system of axioms would ensure that definite properties are uniquely determined by the fundamental relations in every model *in the same way*. This would ensure *non-arbitrariness*, that is, the satisfaction of requirement (d) that figured in the 1908 characterisation, without been fully accounted for. We will return to Zermelo's model-theoretic considerations shortly, when we discuss (Zermelo, 1930c).

The second characterisation of definiteness Zermelo sketches in (Zermelo, 1929) is syntactical. Its purpose is to further clarify which properties are definite. Zermelo asks:

> But which propositions and properties are now in fact 'definite'? How can we decide whether a given proposition is 'definite'?

To answer these questions Zermelo first considers a genetic characterisation of definiteness (Zermelo, 1929, p. 363; his italics):

> *A proposition is called "definite" for a given system if it is constructed from the fundamental relations of the system only by virtue of the logical elementary operations of negation, conjunction and disjunction, as well as quantification, all these operations in arbitrary yet finite repetition and composition.*

He finds a genetic characterisation wanting for the reasons already mentioned, in particular, for its relying on the natural number concept. He therefore presents his "axiomatic approach". As in (Zermelo, 1908), Zermelo starts by presupposing a given a domain, B, and a system of *fundamental relations* over B. He then gives the following clauses (Zermelo, 1929, p. 365):

I) all fundamental relations are definite;

II) 1) the logical operations of negation preserves definiteness;

2) conjunction and disjunction preserve definiteness;

3) the first-order quantifiers preserve definiteness;

4) the second-order quantifiers *applied to "definite functors"* preserve definiteness. More precisely, if $F(f)$ is definite for all *definite* "functors" f, then so is the quantified statement $\forall f F(f)$.

Zermelo supplements these clauses with a closure condition, whose aim is to rule out non-definite properties without making use of the natural number concept. Calling P the totality of all definite properties, he states that no subtotality of P satisfies all the postulates I) and II) (Zermelo, 1929, p. 364).[22]

The fourth clause is particularly important, as it extends the notion of definiteness to the second-order (although it introduces also a puzzling restriction, by requiring that the arguments of the second-order quantifiers be definite). It is here that Zermelo clearly goes beyond the first-order characterisation of definability and considers definite also statements involving the second order quantifiers. He will further explore the fundamental shift to the second-order language (unrestricted) in subsequent work, especially (Zermelo, 1930c).

We have seen that Zermelo sketches both a model-theoretic and a syntactic explication of definiteness. In both cases the fundamental relations play a key role in determining what is and what is not definite. This corresponds to component (a) of definiteness in (Zermelo, 1908). Since the fundamental relations are

22 This new definition was criticised by Skolem (1930), who offered two main criticisms: it requires set theory (to express the closure condition) and it uses a vague notion of second order quantification.

presumably determined by the axioms of the theory, condition (b) also plays a role. Furthermore, in both explications the laws of logic, namely component (c), are key to the preservation of definiteness from the fundamental relations to complex ones. Although Zermelo's focus on the decidability of definite properties makes these new explications of definiteness still close to intensional definiteness, there are also new elements to them that point in a different direction. In the model-theoretic case, Zermelo takes a "global" perspective, by requiring intensional definiteness in every model, while in the syntactic case he stresses its preservation with respect to the quantifiers.[23] The model-theoretic characterisation gestured at in 1929 was further explored by Zermelo in the 1930s, reaching a more refined form in (Zermelo, 1930c). There, we argue, we find clear elements of extensional definiteness.

In (1930c) Zermelo presents a second-order set theory.[24] Definiteness is somehow built in and does not figure any more in the separation axiom, similarly to the standard first-order approach to separation in ZFC. Namely, it is the *second-order* language of set theory, alone, that suffices to formulate separation. Zermelo now states that the propositional function used in the separation schema to separate a subset of a given set should be *completely arbitrary*.[25]

In this paper Zermelo is concerned with *normal domains*, which are models of his set theory indexed by inaccessible cardinals. Zermelo crucially presupposes the availability of an unbounded series of (strongly) inaccessible cardinals, which ensures that the totality of normal domains is itself open-ended. As each normal domain is followed by a "next" normal domain, it is to be thought of as *closed* from the point of view of the next normal domain. The interaction between normal domains, which are thought of as closed, and their open-ended hierarchy is taken by Zermelo to offer a satisfactory solution to the antinomies of set theory. The hierarchy of normal domains also brings Zermelo to realise that his set theory is non-categorical, although its models satisfy important isomorphism theorems. In (1930a, p. 437), Zermelo summarises his findings as follows:

> Two normal domains are "isomorphic" if and only if 1) their "bases" (that is, the totality of their urelements) are equivalent to one another and 2) their "characteristics" (that is the upper limits of the occurring alephs) are equal, if, in other words, to each set of one domain there corresponds at least one equivalent one in the other domain. Of two domains with equivalent bases (but different characteristics) one is always isomorphic to a "canonical" development of the other.

A little later, Zermelo mentions a connection between his notion of normal domain and Cantor's "concept of set". He introduces the notion of "closed domain", which, he claims, corresponds to that of set in Cantor's sense. Zermelo claims that a "closed domain" can be reduced to the concept of "categorical system of postulates" and that every normal domain is a closed domain and can therefore be conceived of as a set in a higher normal domain. A more precise characterisation of closed domains is offered in (Zermelo, 1930a, p. 453) as follows:

> A "closed domain" is one which can be determined or ordered by means of a *categorical system of postulates*. It is precisely that which Cantor really meant by his well-known definition of "set".

A closed domain is contrasted by Zermelo with an open domain (Zermelo, 1930a, p. 453):

> An "open domain" is a *well-ordered sequence of domains successively comprising one another* constituted so that *every closed subdomain can always still be extended in it.* [...] Furthermore, the *entire open domain* can be *well-ordered* so that all elements of a *preceding layer* precede all elements of every subsequent one.

23 The syntactic characterisation has important similarities, as well as differences, with Weyl's notion of extensional definiteness that is the focus of the next section.

24 Zermelo's variant of ZF includes second-order formulations of separation and replacement. Zermelo omits infinity, presupposes the axiom of choice, which he considers a "general logical principle" (Zermelo, 1930c, p. 405), and introduces the axiom of foundation. He also assumes urelements.

25 See (Zermelo, 1930c, p. 403, fn. 2), as well as (Zermelo, 1930b, p. 449).

Going back to our distinction between extensional and intensional definiteness, we argue that a domain being closed is a matter of extensional definiteness: we are interested in all the elements of the domain, rather than considering them one by one. It is thus tempting to regard Zermelo's closed domains as completed or "finished", in Cantor sense: as they admit no further extension their elements coexist. A key novelty of Zermelo compared with Cantor is that he develops an axiomatic approach to set theory. He therefore can take models of the axioms of set theory into consideration. This brings him to suggest that a closed domain should be determined by a categorical system of postulates. Therefore, what comprises it, its elements, is fully determinate and fixed in every model in the same way.

To summarize, in the new phase of Zermelo's reflection on definiteness beginning with (Zermelo, 1929), second-order logic and categoricity play a key role. Zermelo sketches both syntactic and model-theoretic presentations of definiteness. The syntactic presentation takes the form of preservation of definiteness through the logical operations, including the second-order quantifiers. The model-theoretic characterisation is given in terms of what is decided within (categorical) models of the axioms of set theory. This subsequently gives rise to a notion of closed domain, which explicates (what we would call) a form of extensional definiteness. It is tempting to think that this notion of definiteness is close to the Cantorian notion of completability. Zermelo, at any rate, seems to have thought that through a crucial use of the axioms of set theory, the notion of closed domain offered a more precise rendering of Cantor's ideas.[26]

4. Weyl

Let us return to the problem of clarifying the notion of a definite property that figures in Zermelo's original 1908 statement of the axiom of Separation. The usual explication—as a property definable in the language of first-order set theory—can be challenged. As we have seen, Zermelo contested one direction, arguing that properties definable in the language of *second-order* set theory can also be definite. We will now see that Weyl comes to reject the reverse direction, that is, to deny that every formula of first-order set theory defines a definite property.

In his Habilitation Vortrag from 1910 Weyl discusses Zermelo's separation schema, writing that:

> According to Zermelo, a definite expression is one whose application or non-application can be determined unequivocally and without arbitrariness on the basis of the fundamental relation ε that holds between the objects of set theory. Here, to my mind, greater precision is necessary because the expression 'determined unequivocally and without arbitrariness' strikes me as too vague. (Weyl, 1910, p. 304; our translation)

He proposes to characterize a definite relation as one obtained from the fundamental relations of set theory, namely membership and equality, by finitely many applications of five principles of definition that he sets forth. These principles have a clear algebraic character but are usually taken to capture the relations that are definable in (what we call) the language of first-order set theory.

Weyl returns to this issue in his pioneering work on predicativity, *Das Kontinuum* (Weyl, 1918), where he mentions that his "investigation began with an examination of Zermelo's axioms for set theory". He now sets out in full detail the principles of definitions in terms of the ordinary logical operations, but crucially he focuses on the case where these operations are applied to the domain of natural numbers. As we will see, the most important change is that Weyl now rejects the idea that quantification over all sets can be assumed to result in a formula that is (intensionally) definite. This rejection springs from a stronger preference for some generative approach to mathematical objects. Indeed, Weyl espouses a view that is closer to Aristotle's austere potentialism than to Cantor's extremely relaxed set-theoretic potentialism. For whereas Cantor accepts a plethora of transfinite sets as completable, Weyl regards every infinity—even that of the natural numbers—as incompletable; for " 'inexhaustibility' is essential to the infinite" (Weyl, 1918, p. 23). Weyl's potentialist outlook becomes particularly clear in a later passage, commenting on how to place mathematics on a sound foundation:

[26] See, e.g., (Ebbinghaus, 2003; Taylor, 2002) for detailed analysis of this phase of Zermelo's foundational reflection. See also (Taylor, 2002) for discussion of another proposal by Zermelo, involving infinitary languages.

> The deepest root of the trouble lies elsewhere: a field of possibilities open into infinity has been mistaken for a closed realm of things existing in themselves. As Brouwer pointed out, this is a fallacy, the Fall and Original Sin of set-theory, even if no paradoxes result from it. (Weyl, 1949, p. 234)

So, for Weyl, every infinite mathematical domain is incompletable.

Weyl proceeds to make an important and highly innovative distinction. Not all infinite (and therefore incompletable) domains have the same character: some are "extensionally determinate", whereas others are not (Crosilla and Linnebo, 2024).[27] Let us follow Weyl and begin with a loose and intuitive way of drawing the distinction, which we will gradually make more precise. A concept's being "clearly and unambiguously defined", Weyl contends,

> does not imply that this concept is extensionally determinate, i.e., that it is meaningful to speak of the *existent* objects falling under it as an ideally closed aggregate which is intrinsically determined and demarcated. (Weyl, 1919, p. 109)

Thus, for an infinite domain to be extensionally determinate is, loosely speaking, for it to be properly demarcated.

Weyl's mathematical genius is revealed when he proceeds to give this loose and intuitive idea a precise logical articulation. His first step in this direction reads as follows.

> Suppose P is a property pertinent to the objects falling under a concept C. [...] if the concept C is extensionally determinate, then not only the question "Does a have the property P?" [...] but also the existential question "*Is there* an object falling under C which has the property P?", possesses a sense which is intrinsically clear. (ibid.)

Let us paraphrase. Suppose some property P is well-defined on all objects falling under a concept C. That is, suppose that for every individual instance a of the concept C, the question whether a has P has an "intrinsically clear sense". (In our terminology, we suppose that P is intensionally definite on instances of C.) What is it, then, for the concept C to be extensionally determinate or "properly demarcated"? Weyl proposes that this demarcation would enable us to quantify over all Cs and ask—again with an intrinsically clear sense—whether there is some C that has P.[28] This suggests the following analysis:

> A concept C is *extensionally determinate* just in case: for every property P such that the question 'Pa?' has an intrinsically clear sense whenever a is C, also the quantificational question '$(\exists x : Cx)Px$?' has an intrinsic clear sense.

In short, for a concept C to be extensionally determinate is for quantification over Cs to preserve the property of possessing an intrinsically clear sense.

The next step is to clarify what is it for a statement to "possess a sense which is intrinsically clear". Weyl writes that a question has an intrinsically clear sense when it "address[es] an existing state of affairs that allows one to answer the question with yes or no" (Weyl, 1921, p. 88). That is, a statement has an intrinsically clear sense when it is subject to the principle of bivalence, which says that the statement is either true or false.

A final step is now very natural—although it is not, as far as we know, explicitly taken by Weyl himself. The idea is to explicate bivalence in the metalanguage with the Law of Excluded Middle holding in the object language. When this step is taken, we arrive at a formal analysis in the object language itself:

27 As a terminological convention, we reserve the term 'extensionally determinate' and its cognates for Weyl's particular explication of the broader idea of extensional definiteness.

28 In a potentialist setting where objects are successively defined and the principles of definition may themselves be an "open system" to which we can make additions, possession of an "intrinsically clear sense" cannot be taken for granted. See (Weyl, 1918, p. 87), as well as (Linnebo and Shapiro, 2023, pp. 2–4).

Extensional determinateness (formal analysis)
A concept C is *extensionally determinate* iff quantification restricted to C preserves the property of LEM holding, that is, iff, for every property P:

$$(\forall x : Cx)(Px \vee \neg Px) \rightarrow (\exists x : Cx)Px \vee \neg(\exists x : Cx)Px$$

We have thus arrived at an idea that Feferman has recently expressed with pleasing succinctness: "What's definite is the domain of classical logic, what's not is that of intuitionistic logic" (Feferman, 2011, 23). To highlight the central role of quantification in this analysis, we prefer to say that a concept that is extensionally determinate in this sense *defines a domain of classical quantification*.

Equipped with this distinction between two kinds of incompletable domains—those that are extensionally determinate and those that are not—we can ask which domains fall on which side. The following passage summarizes Weyl's own view:

> The intuition of iteration assures us that the concept "natural number" is extensionally determinate. [...] However, the universal concept "object" is not extensionally determinate—nor is the concept "property," nor even just "property of natural number". (Weyl, 1919, p. 110)

We see that Weyl is very sparing in what he regards as extensionally determinate. The domain of natural numbers is extensionally determinate, thanks to our "intuition of iteration".[29] But Weyl is unwilling to go much further. Extensional determinateness is lost as soon as we consider properties—or, for that matter, sets—of natural numbers. We will shortly explain why Weyl held this very strict view.

First, though, we wish to observe that the view explains why Weyl came to reject the idea that every formula of (what we would now call) first-order set theory is intensionally definite. To begin, since the concept of natural number is extensionally determinate, there is a set of all natural numbers. And this set obviously has all kinds of subsets. A formula of first-order set theory aspires to quantify over all these subsets—and many other sets as well. But according to Weyl, already the collection of sets of natural numbers fails to be extensionally determinate. Thus, a formula of first-order set theory may quantify over a domain that fails to be extensionally determinate. It follows that such a formula may lack an intrinsically clear sense, or, in our terminology, that it may fail to be intensionally definite. Weyl therefore has to reject the standard explication of Zermelo's 1908 notion of a 'definite property' as being too lax. For this explication allows instances of Separation involving conditions that cannot be guaranteed to be intensionally definite.

Why, then, did Weyl reject the prevailing view that there is a properly demarcated domain of sets of natural numbers and perhaps larger domains still? The answer is that Weyl rejects the combinatorial conception of set as applied to infinite domains:

> The notion of an infinite set as a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and surveyed as a whole by consciousness, is nonsensical: "inexhaustibility" is essential to the infinite. (Weyl, 1918, p. 23)

Without the combinatorial notion of an arbitrary subset of the natural numbers, we are left without a reason to take the domain of all such sets to be extensionally determinate.

Weyl is not content with these negative claims. Starting in 1918, he develops a positive alternative, namely, a novel *predicative* conception of set, which can legitimately be applied to the domain of natural numbers. Since an infinite set is incompletable (or "inexhaustible"), it needs to be described by means of a rule that "indicates properties which apply to the elements of the set and to no other objects". (Weyl, 1918, p. 20) And these rules need to be carefully specified in a bottom-up manner that avoids any circularity. In essence, when formulating a rule that determines membership in an infinite set, it is permissible to quantify over the natural

[29] (Hartimo, forthcoming) suggests that Husserl's notion of "material definiteness" is rather like Weyl's "extensional determinateness", and that the former's "formal definiteness" is like Zermelo's "categorically determined". The idea is that what ensures that a domain is extensionally determinate, for Weyl, is that the domain is "materially" generated from below in some iterative procedure.

numbers, since these are independently given; but it is not permissible to quantify over sets of numbers, since these are the very objects we are trying to characterize.

Suppose we follow Weyl and allow only predicative subsets of $\mathbb{N}$. Then there is a strong argument that the collection of sets of naturals is *not* properly demarcated or extensionally determinate. For, if this collection were extensionally determinate, we could use quantification over the collection to define yet further sets of numbers. It follows, therefore, that there is no extensionally determinate collection of absolutely all sets of natural numbers.

Let us examine what this view yields when we explicate extensional determinateness in the way outlined above, that is, using intuitionistic logic. Every atomic predication of natural numbers is *decidable*. For example, we have:

$$x + y = z \vee \neg\, x + y = z$$

Likewise, since the domain of natural numbers is assumed to be extensionally determinate, quantification over this domain behaves classically. We capture that by means of the so-called principle of *Bounded Omniscience (BOM)*:

$$(\forall x : \mathbb{N}(x))(\varphi(x) \vee \neg\varphi(x)) \rightarrow (\exists x : \mathbb{N}(x))\varphi(x) \vee \neg(\exists x : \mathbb{N}(x))\varphi(x)$$

An easy induction on syntactic complexity now shows that LEM holds for every formula of first-order arithmetic. By contrast, we do *not* have Omniscience for sets of naturals. That is, the quantifier '$\forall X \subseteq \mathbb{N}$' behaves intuitionistically, not classically.

5. Three notions of definiteness

Let us take stock. We hope to have made it clear that definiteness isn't just a "philosophers' notion", detached from "real" mathematics. Some leading mathematicians during the first half century of modern set theory thought long and hard about definiteness. They typically started with some intuitive but rather loose ideas. That is, they started in what one may regard as philosophical territory. Through a mixture of philosophical and mathematical analysis, these intuitive ideas were then gradually transformed to precise logical notions.

Moreover, it has transpired that several different notions of definiteness are in play. There are two main families. First, a condition can be *intensionally definite*, in the sense of being well-defined or sharply defined in an instance-by-instance manner. This idea figures in early Cantor as well as in (Zermelo, 1908) and the ensuing debate. The idea is naturally explicated in terms of bivalence (as in Weyl, 1921, p. 88)—and thus also, in the context of intuitionistic logic, in terms of LEM (as we have suggested). Then, there is *extensional definiteness*, which is a collective property of a collection of objects, not a distributive one. We have seen that this notion admits of two distinct explications. One is in terms of the potentialist idea of possible coexistence or completability, which we encountered in Cantor (especially in his letters to Hilbert and Dedekind) but also in Zermelo's later work, such as (Zermelo, 1930). Weyl proposes a different explication. He starts with the intuitive idea of a domain being properly demarcated, which he explicates as a matter of the domain supporting classical quantification.

The following table summarizes our findings, with intuitive philosophical ideas in italics and sharp logico-mathematical explications in boldface.

ID		**ED**	
sharp, well-defined			*completable*
Cantor			Cantor
↓			↓
Zermelo '08			Zermelo '30
↓			
Weyl	→	*properly demarcated* Weyl	↓
↓		↓	
decidable		**classical quantification**	a plurality

To recall, a concept C is said to be *decidable* just in case $\forall x(Cx \vee \neg Cx)$. Further, C is said to define *a domain of classical quantification* just in case C is decidable and for every property P:

$$(\forall x : Cx)(Px \vee \neg Px) \rightarrow (\exists x : Cx)Px \vee \neg(\exists x : Cx)Px$$

It remains only to clarify the precise logico-mathematical cash value of the domain forming a plurality.

Before we do that, though, we would like to make two initial observations about the relation between the two forms of extensional definiteness. First, Weyl's notion of extensional determinateness can be less demanding than Cantor's neo-Aristotelian notion of completability. We have seen that Weyl regards some domains, such as that of the natural numbers, as incompletable yet still extensionally determinate. While such domains have a well-defined extension, they simultaneously have an irreducibly intensional character. The incompletability means that the domain cannot be specified as a plurality of objects but instead requires an intensional specification. Second, Weyl's notion of extensional determinateness is more closely connected with the notion of intensional definiteness than Cantor's neo-Aristotelian notion of completability. For a domain to be extensionally determinate, in Weyl's sense, just is for quantification over the domain to preserve intensional definiteness. This explains the single left-to-right arrow in the above table.

We turn now to the task of providing a sharper description of the logical-mathematical cash value of having a plurality of objects as opposed to "just" an extensionally determinate domain. Part of the answer is found in [an already quoted] passage from Weyl:

> The notion of an infinite set as a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and surveyed as a whole by consciousness, is nonsensical: "inexhaustibility" is essential to the infinite. (Weyl, 1918, 23)

Weyl claims that when a domain is incompletable (or "inexhaustible"), the combinatorial conception of an arbitrary subset of the domain is unavailable. He also implies that this is *the reason* why the combinatorial conception is unavailable, which suggests that the availability of the combinatorial conception goes hand in hand with the completability of the domain. We believe this is correct—even if one follows Cantor, as against Weyl (and Aristotle) and regards some infinite domains as completable. The completability of a domain, whether finite or infinite, licences talk about arbitrary subsets of the domain.

We would like to extend this analysis yet further. To do so, we find it useful to recall (Bernays, 1935)'s rightly famous notion of "quasi-combinatorial" reasoning. The idea is introduced in the following passage:

> But analysis is not content with this modest variety of platonism [i.e. the use of classical logic]; it reflects it to a stronger degree with respect to the following notions: set of numbers, sequence of numbers, and function. It abstracts from the possibility of giving definitions of sets, sequences, and functions. These notions are used in a "quasi-combinatorial" sense, by which I mean: in the sense of an analogy of the infinite to the finite. (p. 259)

To reason quasi-combinatorially is thus to treat an infinite domain as if it were finite. This "analogy of the infinite to the finite" has several aspects. First, quantification over the domain can be understood as infinite conjunctions or disjunctions of instances. Second, we argue as if we can form arbitrary sets of objects from the domain. Thus, in particular, impredicative comprehension is permissible for sets of objects from the domain. Relatedly, the Axiom of Choice will be acceptable. For any family of non-empty and non-overlapping sets from the domain, there is a choice set containing precisely one member of each of the mentioned sets.

We can now state our proposal. *The logical-mathematical cash value of a domain being completable, or specifiable as a plurality, is that quasi-combinatorial reasoning about the domain is licensed.*[30] Plugging this analysis into the above table and compressing the chronology, we obtain the following truncated table:

[30] See (Florio and Linnebo, 2021, Chs. 10 and 12) for an elaboration and defense of this proposal. As the authors acknowledge (pp. 288-89), this explication means that plural logic has rich and strong mathematical content. Indeed, as we have seen, Weyl (1918, p. 23) finds the idea of an infinite plurality, in this sense, "nonsensical".

ID	ED	
sharp, well-defined	*properly demarcated*	*completable*
Cantor, Zermelo, Weyl	Weyl	Cantor, Zermelo
decidable	**classical quantification**	**quasi-combinatorial**

We observed above that a concept can define a domain of classical quantification without defining a quasi-combinatorial domain. We now contend that there is an inclusion in the reverse direction; that is, every quasi-combinatorial domain is also a domain of classical quantification. When a domain can be completed as a plurality of objects, say *dd*, then for any generalization over the domain, there is a well-defined plurality of instances concerned with objects from the domain. This enables us to understand quantification over the domain as a (perhaps infinite) conjunction or disjunction of instances. And this, in turn, ensures that the domain can be taken to support classical quantification. Thus, the Cantorian notion of extensional definiteness entails Weyl's. It follows that Cantor's notion is strictly more demanding than Weyl's.

6. A classification of views in the foundations of mathematics

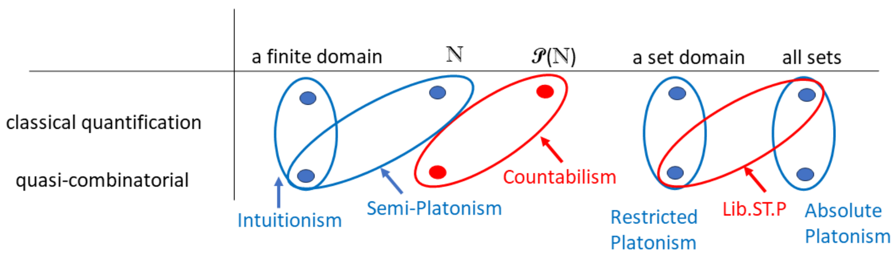
We wish to close by using the two forms of extensional definiteness—namely, defining a domain of classical quantification or even a quasi-combinatorial domain—to describe a way to classify various views in the foundations of mathematics. A key question to ask of any view in the foundations of mathematics, we propose, is how large a domain its proponents are willing to regard as extensionally definite in either of these two senses.

The most important types of domain to consider are: finite domains, the domain of all natural numbers, the domain of all sets of natural numbers, any domain based on a Cantorian transfinite set, and the domain of all Cantorian sets. Thus, there are five types of domain, each of which can be classified as having either a weak or a strong form of extensional definiteness or neither of the two. We propose to measure the theoretical commitments of a view in the foundations of mathematics by asking how strong a form of extensional definiteness the view is willing to ascribe to how large a domain. Bernays (1935) famously regards these theoretical commitments as steps towards platonism in the foundations of mathematics.

To illustrate how our proposed classification works, let us apply it to four important positions in the foundations of mathematics identified by Bernays (1935), whose associated theoretical commitments he orders as follows:[31]

$$\text{intuitionism} < \text{“semi-platonism”} < \text{“restricted platonism”} < \text{“absolute platonism”}$$

First, intuitionism accepts only finite domains as extensionally definite in either respect. Next, "semi-platonism" regards the collection of natural numbers as a domain of classical quantification but not as a quasi-combinatorial domain (pp. 263 and 268). This is predicativism in the sense of (Weyl, 1918). Then, "restricted platonism", which is Bernays' own preferred view, regards any Cantorian transfinite set as completable (and *a fortiori* also a classical domain) (p. 261). Lastly, there is "absolute platonism", which regards the entire universe of sets as completable (pp. 261, 267, and 269). According to Bernays, absolute platonism is shown incoherent by the set-theoretic paradoxes, while "restricted platonism is not touched at all by the antinomies" (p. 261; see also p. 269). We thus obtain the following classification:



31 Although (Bernays, 1935) does not use the word 'definite' or any cognate thereof, the four positions he identifies consist of answers to the question of which domains have the two forms of extensional definiteness that we have identified.

The classification can be extended beyond Bernays' four positions as well. Let us briefly mention two examples. First, we believe there is a coherent and interesting position that regards the countable infinity of natural numbers as completable, and the collection of all sets of natural numbers as a classical domain, but that refuses to go further with either notion of definiteness. This position is a form of *countabilism*.[32] Second, it is interesting to ask whether there is a coherent position that regards the collection of all transfinite sets as a domain of classical quantification but not as a quasi-combinatorial domain. In the literature on potentialism, this view has had several defenders, sometimes under the name of "liberal set-theoretic potentialism".[33] Others, though, have questioned the coherence of the view, arguing that in the end it fares no better than Bernays' "absolute platonism".

We submit that the two notions of extensional definiteness that emerge from our investigation enable us to identify and understand some of the most important fault lines in the philosophy and foundations of mathematics. Thus, far from being detached philosophy, these notions are mathematically illuminating.

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[32] See, e.g. (Scambler, 2021), (Builes and Wilson, 2022), and (Rathjen, 2016).

[33] See, e.g. (Linnebo, 2013), (Studd, 2013), and (Linnebo and Shapiro, 2019).

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