Aristotle meets Frege: from Potentialism to Frege Arithmetic

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Abstract: The purpose of this paper is to present a genuinely potentialist account of Frege arithmetic. The (cardinal) numbers are not generated from Hume's Principle, but rather from more or less standard principles of potentialism. The relevant version of Hume's Principle is a principle stating a condition for numbers to be identical with each other. Essentially, (HP) tells us what we are generating—cardinal numbers—but the generation does not go through (HP) itself. We also develop an Aristotelian, potentialist set theory—in effect, a theory of hereditarily finite sets—a theory that is definitionally equivalent to Dedekind-Peano arithmetic.

Keywords: Abstractionist neo-logicism, Crispin Wright, Bob Hale, Hume's Principle, Aristotelian potentialism, finitude, Dedekind finite

1. Introduction

The abstractionist, neo-logicist program in the philosophy of mathematics began with Crispin Wright's seminal (Wright, 1983). Bob Hale (1983) joined the cause, and it continues through many extensions, objections, and replies to objections (see Hale and Wright, 2001). The program's overall plan is to develop branches of established mathematics using abstraction principles in the form:

$$\forall a \forall b (\Sigma(a) = \Sigma(b) \leftrightarrow E(a,b)), \tag{ABS}$$

where a and b are variables of a given type (typically first-order or monadic second-order), Σ is an operator, denoting a function from items of the given type to objects in the range of the first-order variables, and E is an equivalence relation over items of the given type.

Gottlob Frege (1884, 1893), employed at least three equations in the form (ABS). One of them, used for illustration, comes from geometry:

> The direction of l_1 is identical to the direction of l_2 if and only if l_1 is parallel to l_2.

The second was dubbed $N^=$ in (Wright, 1983) and is now called *Hume's Principle*:

$$\forall F \forall G (\#F = \#G \leftrightarrow F \approx G), \tag{HP}$$

where $F \approx G$ is an abbreviation of the second-order statement that there is a one-to-one relation mapping the F's onto the G's. In words, (HP) states that the number of F is identical to the number of G if and only if F is

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equinumerous with G. Georg Cantor deployed this principle to obtain extensive and profound results, especially concerning the transfinite.[1]

Unlike the direction-principle, the relevant variables, F, G here are second-order. We will follow the literature and refer to items in the range of these variables as “Fregean concepts”, or sometimes just “concepts”, essentially properties construed extensionally (and not necessarily identified with mental phenomena).

Frege’s (1893) third exemplar of an abstraction principle is the infamous Basic Law V:

$$\forall F \forall G(\varepsilon F = \varepsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx)). \tag{BLV}$$

Like Hume’s Principle, Basic Law V is second-order, but unlike Hume’s Principle, it is inconsistent (at least with classical or intuitionistic logic).

As is now well-known, Frege’s *Grundlagen* (1884) and *Grundgesetze* (1893) contain the essentials of a derivation of the Dedekind-Peano postulates from Hume’s Principle, plus some more or less straightforward definitions.[2] This result, now called *Frege’s Theorem*, reveals that Hume’s Principle, together with suitable definitions, entails that there are infinitely many natural numbers. The development of arithmetic from (HP) is sometimes called *Frege arithmetic*. This theory is taken to be the first success story of abstractionist neo-logicism. The underlying theme is that one can introduce (HP) as a sort of stipulative, implicit definition of the “number-of” operator, and develop arithmetic from that. There is an ongoing program of attempting to found other, richer mathematical theories on abstraction principles. Here, we will only be concerned with arithmetic and (HP).

In their informal discussion of the abstractionist program, Wright and Hale occasionally speak of mathematical objects as “generated” by the abstraction principle (e.g., Hale and Wright, 2001, pp. 19, 224, 237n, 278, 289, 412, 414), but for them, this term is only a metaphor. Their version of abstractionism is not a potentialist enterprise, as the quantifiers in the description of the equivalence relation on the right hand side are explicitly intended to include the “generated” abstracts, cardinal numbers in this case. Frege’s Theorem depends on this. Sometimes, the word “generated” appears in scare-quotes, as in:

> One obvious danger here arises from the fact an equivalence relation defined on the concepts on a specified underlying domain of objects may partition those concepts into more equivalence classes than there are objects in the underlying domain, so that a second-order abstraction may ‘generate’ a domain of abstracts strictly larger than the initial domain of objects. This, in itself, need be no bad thing—indeed, it is essential, if there is to be a neo-Fregean abstractionist route to (classical) analysis. (Hale and Wright, 2001, p.19)

The plan here is to present a genuinely potentialist account of Frege arithmetic. The (cardinal) numbers are not generated from (HP), but rather from more or less standard principles of potentialism. The relevant version of (HP) is a principle stating a condition for numbers to be identical with each other. So the account is not an abstractionist one. Essentially, (HP) tells us what we are generating—cardinal numbers—but the generation does not go through (HP) itself.

The perspective here is that of an Aristotelian potentialist who rejects even the possibility of an actual infinity (as presented in Linnebo and Shapiro, 2019). We also develop an Aristotelian, potentialist set theory—in effect, a theory of hereditarily finite sets—a theory that is definitionally equivalent to Dedekind-Peano arithmetic. The orientation is deductive: we articulate an axiomatic (higher-order and plural) language in which to express the main principles, and explore what can be deduced in order to express and sustain the potentialist insights (if that is what they are).

This is in contrast with the lovely study Stafford (2023), which draws on Hodes (1990). Like the present project, Stafford treats potentialist arithmetic, drawing on (HP), but its orientation is “semantic”, i.e., model-theoretic. Using a background set theory, presumably full ZFC (or perhaps Zermelo set theory), he develops

[1] More details to follow. By rights, this abstraction principle should be called “Cantor’s Principle”, but the name “Hume’s Principle” has caught on.

[2] Frege (1893) used Basic Law V to derive the two conditionals in (HP). The rest of the Dedekind-Peano postulates follow from those.

Kripke structures for a modal language, structures in which each world is finite. He shows how to interpret a standard first-order (but not second-order or plural) version of Dedekind-Peano arithmetic in the model-theoretic structures. This particular model-theoretic perspective is, of course, not available to an Aristotelian, since the meta-theory makes heavy use of actual infinity. In the developed framework, each world is finite, but the Kripke structure itself has infinitely many worlds.

Stafford notes that he leaves the "development of a deductive theory for future work" (p. 557). Although the present project is deductive, it does not recapitulate Stafford's theory. The target here is full second-order Dedekind-Peano arithmetic, but as noted, Stafford's theory does not satisfy that.[3]

The present paper is self-contained. The next Section 2 provides a brief overview of the potentialist perspective—for more details, see (Linnebo and Shapiro, 2019). We present an axiom, called "(Aristotle)", that entails that infinite concepts and pluralities are impossible—all worlds are (Dedekind) finite. Section 3 provides a sketch of Frege's own development of arithmetic (based on Hume's Principle (HP)), giving the usual definitions. Section 4 shows how to formulate (HP) in the modal, potentialist setting, along with axioms entailing the possible existence of various numbers. Then, in Section 5, we formulate definitions and show how to derive the relevant analogues of the Dedekind-Peano axioms. Section 6 drops the (Aristotle) axiom. At that point, we do not assert the possibility of an actual infinity; rather, the theory is officially neutral on whether there is or there could be an actual infinity. The development there is a bit closer to Frege's own treatment, along with that of the abstractionist neo-logicists. The final Section 7 presents an Aristotelian set theory (of hereditarily finite sets), and shows that arithmetic can be interpreted in that theory in the usual way.

2. Potential infinity — a crash course

Aristotle famously rejected the actual infinite—the existence of any complete collection with infinitely many members. He argued that the only sensible notion is that of potential infinity. In *Physics* 3.6 (206a27-29), he wrote:

> For generally the infinite is as follows: there is always another and another to be taken. And the thing taken will always be finite, but always different (2o6a27-29).

As Richard Sorabji (2006, pp. 322-3) once put it, for Aristotle, "infinity is an extended finitude"(see also Lear, 1980). This attitude toward the infinite was expressed by the vast majority of mathematicians and philosophers at least until late in the nineteenth century. In 1831, for example, Gauss (1831) wrote:

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking.

In line with the mathematicians of his day, Aristotle did accept what is sometimes called *potential infinity*, against the ancient atomists (see Miller, 2014). Mathematicians in antiquity followed this, and, indeed, made brilliant use of potential infinity. But what is potential infinity?

Either directly or indirectly, the idea seems to be that potential infinity is tied to certain *procedures* that can be repeated indefinitely. A nice example is provided by Aristotle's claim, against the atomists, that matter is infinitely divisible. Consider a body of mud. However many times one has divided the mud, it is always possible to divide it again—or so it is assumed.

As indicated by the term "it is possible", the thesis here can be explicated in a modal way.[4] This yields the following analysis of the infinite divisibility of the body of mud s:

$$\Box\forall x(Pxs \to \Diamond\exists yPyx), \tag{6}$$

[3] Thanks to a referee for pressing this comparison, and to Tim Button for an insightful exchange on the issues.

[4] I make use of contemporary modal notions here. There is no attempt to recapitulate what Aristotle himself says about modality.

where the variables range over parts of the given body of mud, and Pxy means that x is a *proper* part[5] of y. If the parts of the mud formed an actual infinity, the following would hold:

$$\forall x(Pxs \to \exists y Pyx). \tag{7}$$

According to Aristotle, it is impossible for there to be infinitely many divisions of the mud, existing all at once:

$$\neg\Diamond\forall x(Pxs \to \exists y Pyx). \tag{8}$$

By endorsing both (6) and (8), one is is asserting that the divisions of the mud are *merely potentially infinite*.

As noted, present concern is with mathematics, and the natural numbers in particular. According an Aristotelian, the sequence of natural numbers is merely potentially infinite. This can be represented as the conjunction of the following theses:

$$\Box\forall m\Diamond\exists n\,\textsc{Succ}(n,m) \tag{9}$$

$$\neg\Diamond\forall m\exists n\,\textsc{Succ}(n,m), \tag{10}$$

where $\textsc{Succ}(n,m)$ states that n comes right after m. The modal language thus provides a nice way to distinguish the merely potential infinite from the actual infinite.

Linnebo and Shapiro (2019) develop an account that can accept some actually infinite collections, and still leaves room to insist that some other "totalities" are merely potentially infinite. The analysis provides a framework in which actual and potential infinity can live side by side, sometimes in the very same system. Here the focus is on a more Aristotelian perspective that allows no actually infinite collections. Our goal is to vindicate, for arithmetic,[6] Aristotle's claim about geometry:

> Our account does not rob the mathematicians of their study, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the <actually> infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. (207b27-30)

2.1. Three orientations towards the infinite

It is useful to distinguish different orientations towards a given infinite totality, such as the natural numbers or the parts of a given body of mud (according to Aristotle). *Actualism* accepts actual infinities, of the given kind, and thus finds no use for modal notions—or at least no use that is specific to the analysis of the infinity in question. Actualists maintain that the non-modal language of ordinary mathematics is already fully explicit and thus deny that we need a translation into some modal language. Furthermore, actualists accept classical logic when reasoning about the infinite (or the infinite in question).

Potentialism is the orientation that stands opposed to actualism. Accordingly, the objects with which mathematics is concerned—or some of the objects with which mathematics is concerned—are generated successively, and at least some of these generative processes cannot be completed. Present concern is with the natural numbers. Our potentialist thinks of numbers as generated, presumably one at a time.

There are (at least) two different forms of potentialism. As characterized above, potentialism is the view that some or, in the present case, all of the objects with which mathematics is concerned are successively generated. What about *the truths* of mathematics? Of course, on any form of potentialism, these are modal truths concerned with certain generative processes. But how should these modal truths be understood?

Liberal potentialists regard the modal truths as unproblematic, adopting bivalence for the modal language, and excluded middle for the underlying modal logic. Consider Goldbach's conjecture. As potentialists interpret

5 A referee notes that this presupposes that the "parts" of the mud all have non-zero measure. In particular, the mud that occupies an extensionless point would have no proper parts, thus violating (1). However, Aristotle explicitly insisted that points are not parts of anything—see, for example, (Hellman and Shapiro, 2018, Chapter 1).

6 There is some anachronism here. As far as we know, there was nothing like full Dedekind-Peano arithmetic in Aristotle's day.

it, the conjecture says that necessarily any even natural number, greater than two, that is ever generated can be written as a sum of two primes. Liberal potentialists maintain that this modal statement has a truth-value—it is either true or false. Their approach to modal theorizing in mathematics is thus much like a realist approach to modal theorizing in general: there are objective truths about the relevant modal aspects of reality, and this objectivity warrants the use of some classical form of modal logic.

Strict potentialists differ from liberal potentialists by requiring, not only that every object generated at some stage of a process, but also that every truth be "made true" at some stage. Consider, again, the Goldbach conjecture. If there are counterexamples to the conjecture, then its negation will presumably be "made true" at the stage where the first counterexample is generated. But suppose there are no counterexamples. Since the conjecture is concerned with *all* the natural numbers, it is hard to see how it could be "made true" without completing the generation of natural numbers. This completion would, however, violate the strict potentialists' requirement that any truth be made true at some stage of the process.

Linnebo and Shapiro (2019) argue that strict potentialists should adopt a modal logic whose underlying logic is intuitionistic (or intermediate between classical and intuitionistic logic). This allows them to adopt a conception of universal generality which does not presuppose that all the instances are ever "available" at a stage. In particular, strict potentialists should not accept every instance of the law of excluded middle in the background modal language. For the most part, we adopt the liberal perspective here, at least partly because we wish to recapitulate a version of classical arithmetic.

2.2. The modality and the modal logic

It is useful here to invoke the contemporary heuristic of possible worlds when discussing the modality in question. But it is insisted that this is *only* heuristic, as a manner-of-speaking (unlike the treatment in Stafford, 2023). The theory is formulated in the modal language, with (one or both of) the modal operators as primitive. The modal language is not explained or defined in terms of anything else.

The potentialist does, of course, reject the now common thesis that mathematical objects exist of necessity—if they exist at all. To invoke the heuristic, the now common thesis is that all mathematical objects exist in all worlds. The potentialist gives that up. There is no world with all of the objects in question—all natural numbers in the present case. Nor, of course, is there an actual infinity of possible worlds (again, unlike Stafford, 2023).

What about the philosophical nature of the modality invoked in the analysis of potentiality? For the Aristotelian, it can perhaps be an ordinary metaphysical modality invoked in contemporary philosophy (or perhaps defined from that notion)—waiving the now widely held thesis that mathematical objects exist necessarily if they exist at all. For that matter, the modality can also be a very ordinary circumstantial modality, as studied in linguistic semantics (suitably idealized, of course).

The idea is that natural numbers are generated in time. At any stage—in any world—there are finitely many natural numbers, but each such world has access to another where some more numbers have been generated. Given enough time, any natural number can be generated, even though there is no time when they are all generated.

Following the heuristic, we assume that every possible world is finite, in the sense that it contains only finitely many objects. For convenience, we wish to avoid invoking a free logic, at least here. So we do not wish to countenance objects—mathematical or otherwise—going out of existence. To paraphrase Aristotle, we study generation, but not corruption. This entails that the domains of the possible worlds grow (or, better, are non-decreasing) along the accessibility relation. So we assume:

$$w_1 \leq w_2 \rightarrow D(w_1) \subseteq D(w_2) \tag{11}$$

where '$w_1 \leq w_2$' says that w_2 is accessible from w_1, and for each world w, $D(w)$ is the domain of w.

Again, for convenience, in this initial foray, one can think of a possible world as determined completely by the objects it contains. So we assume the converse of (11). This motivates the following principle:

> **Partial ordering**: The accessibility relation $\leq$ is a partial order. That is, it is reflexive, transitive, and anti-symmetric.

So the underlying modal logic is at least S4. As is well-known, the conditional (11) entails that the converse Barcan formula is valid. That is,

$$\exists x \Diamond \phi(x) \to \Diamond \exists x \varphi(x). \tag{CBF}$$

So far, then, we have S4 plus (CBF).[7]

We also assume, for simplicity (and convenience) that the only things generated are mathematical. If we think of the natural numbers as generated one at a time, in order, it is perhaps natural to assume that the possible worlds have a linear ordering. But it is useful to tie the present framework in with other, more general ones, in order to invoke, or eat least discuss, some existing results. At any stage in a process of construction, we generally have a choice of which objects to generate. For some types of construction, but not all, it makes sense to require that a license to generate objects is not revoked at accessible worlds. Intuitively, geometric construction is like this. For example, we might have, at some stage, two intervals that don't yet have bisections. We can choose to bisect one or the other of them, or perhaps to bisect both simultaneously. Assume we are at a world w_0 where we can choose to generate objects, in different ways, so as to arrive at either w_1 or w_2. Say at w_1 we bisect an interval i and at w_2 we bisect another interval j. It seems plausible to require that the license to bisect i can be executed at w_2 or any later world. In other words, nothing we do can prevent us from being able to bisect the other interval.

This corresponds to a requirement that any two worlds w_1 and w_2 accessible from a common world have a common extension w_3. This is a directedness property known as *convergence* and formalized as follows:

$$\forall w_0 \forall w_1 \forall w_2 (w_0 \leq w_1 \wedge w_0 \leq w_2 \to \exists w_3 (w_1 \leq w_3 \wedge w_2 \leq w_3))$$

For constructions that have this property, then, we adopt the following principle:

> **Convergence**: The accessibility relation $\leq$ is convergent.

This principle ensures that, whenever we have a choice of mathematical objects to generate, the order in which we choose to proceed is irrelevant. Whichever object(s) we choose to generate first, the other(s) can always be generated later. Unless $\leq$ is convergent, our choice whether to extend the ontology of w_0 to that of w_1 or that of w_2 might have an enduring effect.[8]

It is well known that the convergence of $\leq$ ensures the soundness of the following principle:

$$\Diamond \Box p \to \Box \Diamond p. \tag{G}$$

The modal propositional logic that results from adding this principle to a complete axiomatization of S4 is known as S4.2.

2.3. The logic of potential infinity

What is the correct logic when reasoning about potentially infinite collections? Informal glosses aside, the language of mathematics is usually non-modal. We thus need a translation to serve as a bridge connecting the non-modal language in which mathematics is ordinarily formulated with the modal language in which our analysis of potentiality is developed. Suppose we adopt a translation $*$ from a non-modal language $\mathscr{L}$ to a

[7] Recall that S4 and (non-free) first-order logic entails (CBF). We can also require the accessibility relation to be well-founded, on the grounds that all mathematical construction has to start somewhere. However, nothing of substance turns on this here.

[8] Other types of "generation" are not like this. Suppose for example, that I can bake bread or I can bake a cake, since I have enough time and ingredients to do either. But it may be that if I bake bread, then I can no longer bake a cake, since I may have used up the needed ingredients or I won't have enough time. Or suppose that a country has a law that a couple can have only two children. If a given couple already has one, then it is possible for them to have a boy and it is possible for them to have a girl. But if they do one, they will not be allowed to go on and do the other.

corresponding modal language $\mathscr{L}^{\Diamond}$. The question of the right logic of potential infinity is the question of which entailment relations obtain in $\mathscr{L}$.

To determine whether $\varphi_1, \ldots, \varphi_n$ entail ψ, we need to (i) apply the translation and (ii) ask whether $\varphi_1^*, \ldots, \varphi_n^*$ entail ψ^* in the modal system. This means that the right logic of potential infinity depends on several factors. First, the logic depends on the bridge that we choose to connect the non-modal language of ordinary mathematics with the modal language in which our analysis of potential infinity is given. Second, the logic obviously depends on our modal analysis of potential infinity; in particular, on the modal logic that is used in this analysis—in particular, on whether the underlying logic of the modal language is classical or intuitionistic. Let us now turn to the first factor.

The heart of potentialism, or at least the present explication of potentialism, is the idea that the existential quantifier of ordinary non-modal mathematics has an implicit modal aspect. In the developed interpretive program, a statement that a given number has a successor is interpreted as a statement that this number *potentially* has a successor—that it is *possible* to generate a successor. This suggests that the right translation of $\exists$ is $\Diamond\exists$.

Similarly, a potentialist statement that a given property holds of all objects (of a certain sort) is interpreted as a statement that the property holds of all objects (of that sort) *whenever they are generated*. This suggests that $\forall$ be translated as $\Box\forall$.

Thus understood, the quantifiers of ordinary non-modal mathematics are devices for generalizing over absolutely all objects, not only the ones available at some stage, but also any that we may go on to generate. In our modal language, these generalizations are effected by the strings $\Box\forall$ and $\Diamond\exists$. Although these strings are strictly speaking composites of a modal operator and a quantifier proper, they behave logically just like quantifiers ranging over all entities at all (future) worlds.

The proposal is thus that each quantifier of the non-modal language is translated as the corresponding modalized quantifier. Each connective is translated as itself. Let us call this the *potentialist translation*, and let $\varphi^{\Diamond}$ represent the translation of φ. We say that a formula is *fully modalized* just in case all of its quantifiers are modalized. Clearly, the potentialist translation of any non-modal formula is fully modalized.

Say that a formula φ is *stable* if the necessitations of the universal closures of the following two conditionals hold:

$$\varphi \to \Box\varphi \qquad\qquad \neg\varphi \to \Box\neg\varphi$$

This sets the stage for two key results, which answer the question about the correct logic for those kinds of potentiality that enjoy the above convergence property.

For the first, let $\vdash$ be the relation of classical deducibility in a non-modal first-order language $\mathscr{L}$. Let $\mathscr{L}^{\Diamond}$ be the corresponding modal language, and let $\vdash^{\Diamond}$ be deducibility in the modal system consisting of classical $\vdash$, S4.2, and axioms asserting the stability of all atomic predicates of $\mathscr{L}$.

Classical potentialist mirroring. For any formulas $\varphi_1, \ldots, \varphi_n, \psi$ of $\mathscr{L}$, we have:

$$\varphi_1, \ldots, \varphi_n \vdash \psi \quad \text{if and only if} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

(See Linnebo, 2013, for a proof.)

The theorem has a simple moral. Suppose we are interested in logical relations between formulas in the range of the potentialist translation, in a classical, first-order modal theory that includes S4.2 and the stability axioms. Then we may delete all the modal operators and proceed by the ordinary non-modal logic underlying $\vdash$. In particular, under the stated assumptions, the modalized quantifiers $\Box\forall$ and $\Diamond\exists$ behave logically just as ordinary quantifiers, except that they generalize across all (accessible) possible worlds rather than a single world.[9]

[9] Note the restriction to first-order languages. The result will apply to higher-order languages if the modal translations of the instances of the comprehension scheme are deducible in the modal language.

As noted above, however, Linnebo and Shapiro (2019) argue that a stricter form of potentialism pushes in the direction of intuitionistic logic. There is a second mirroring theorem. In a system governed by intuitionistic logic, a formula φ is said to be *decidable* if the universal closure of $\varphi \vee \neg\varphi$ is deducible in that theory. Let $\vdash_{\text{int}}$ be the relation of intuitionistic deducibility in a first-order language $\mathscr{L}$, together with axioms stating the decidability of all atomic formulas, and let $\vdash^{\Diamond}_{\text{int}}$ be deducibility in the modal language corresponding to $\mathscr{L}$, by $\vdash_{\text{int}}$, S4.2 and the stability axioms for all atomic predicates of $\mathscr{L}$.[10] The conclusion is then the same as above:

Intuitionistic potentialist mirroring For any formulas $\varphi_1, \ldots, \varphi_n, \psi$ of $\mathscr{L}$, we have:

$$\varphi_1, \ldots, \varphi_n \vdash_{\text{int}} \psi \quad \text{if and only if} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{\Diamond}_{\text{int}} \psi^{\Diamond}.$$

(See Linnebo and Shapiro, 2019, for a proof.)

Our interest won't always be limited to formulas in the range of the potentialist translation. One can often use the extra expressive resources afforded by the modal language to engage in reasoning that takes us outside of this range. The modal language allows us to look at the subject matter under a finer resolution, which can be turned on and off, according to our needs.

The final order of business for this section is a case in point. We state an axiom that enforces the Aristotelian thesis that—to invoke the heuristic—all worlds are finite. Our choice is that it is necessary that for any Fregean concept X and for any relation R on X that is a one-to-one function on X, R is a surjection on X (i.e., if Xx then x has an R predecessor):

$$\Box\forall R\forall X[\forall x(((Xx \to \exists y\forall z(Xz \wedge Rxz) \leftrightarrow y = z)) \wedge \forall x_1\forall x_2\forall y((Rx_1y \wedge Rx_2y) \to x_1 = x_2)) \\ \to \forall y(Xy \to \exists x(Xx \wedge Rxy)))] \quad \text{(Aristotle)}$$

This entails that all worlds are at least Dedekind finite.

Intuitively, a Fregean concept, or a possible world, is finite if its cardinality is a natural number. To be sure, it would be premature to invoke that here, since we have not yet defined the natural numbers. But we can express a statement that a concept or world is actually finite (and not just Dedekind finite). Let R be any binary relation, Frege (1879) defined the (weak) *ancestral* R^* of R.[11] Using a more contemporary framework the definition is as follows:

$$R^*xy \leftrightarrow_{\text{def}} [\forall X(Xx \wedge (\forall z\forall w((Xz \wedge Rzw) \to Xw))) \to Xy] \quad \text{(Ancestral)}$$

In words, R^*xy holds if y has every Fregean concept that holds of x and is closed under R. In effect, R^*xy holds just in case either $x = y$ or there is a finite sequence $a_1 \ldots a_{n+1}$ such that $a_1 = x, a_{n+1} = y$ and for each m such that $1 \leq m < n$, $R_{a_m a_{m+1}}$.

We can state a principle that asserts, in effect, that all worlds are finite as follows:

$$\Box\exists R(\forall x\forall y_1\forall y_2((Rxy_1 \wedge Rxy_2) \to y_1 = y_2) \wedge \exists x\forall z R^*(x,z) \wedge \exists y\forall z(\neg R(y,z))) \quad \text{(finite)}$$

In words, (finite) says that necessarily (i.e., in every world) there is one-to-one relation R such that (i) there something x such that everything (in that world) is an R-ancestor of x, and (ii) there is something z that does not bear R to anything (in that world). Intuitively, it follows that in every world, every concept is actually finite in that world. This principle is probably closer to Fregean concerns, and it allows many of the proofs to be constructive.

Nevertheless, we shall stick with the weaker (Aristotle) here. After some other axioms are added, we can derive (finite), showing that all worlds and thus all Fregean concepts are actually finite. It follows that, necessarily (i.e., in every world w), there could be a natural number (possibly in an accessible world w') that is the number of objects (in w).

10 A referee points out that identity is not decidable in intuitionistic analysis (or in smooth infinitesimal analysis) and membership is not decidable in some intuitionistic set theories. The relevant moral is that this mirroring theorem does not hold for those theories.

11 From here on, when we speak of the ancestral or an ancestor, we mean the weak ancestral and a weak ancestor.

3. A sketch of Frege's Theorem

Recall the ill-named Hume's Principle:

$$\forall F \forall G(\#F = \#G \leftrightarrow F \approx G), \tag{HP}$$

where $F \approx G$ is an abbreviation of the second-order statement that there is a one-to-one relation mapping the F's onto the G's. That is:

$$F \approx G \leftrightarrow_{\text{def}} \exists R[\forall x(Fx \to \exists! y(Gy \wedge Rxy)) \wedge \forall y(Gy \to \exists! x(Xx \wedge Rxy))]$$

A central component of (this phase of) the abstractionist program is to establish Frege's Theorem: a derivation of the Dedekind-Peano axioms from (HP) and reasonable definitions of the primitive arithmetical vocabulary. This, it is argued, provides a logical and epistemological foundation for arithmetic. The purpose of this section is to provide a sketch of Frege's Theorem, or at least of some key steps along the way. Then we turn to a potentialist version of the result.

The abstractionist does not presuppose, at the outset, just as a matter of syntax and semantics, that every Fregean concept F has a (unique) number $\#F$. So the proper background would be a free logic. Their practice is to start with a concept F, note the trivial consequence that $F \approx F$, and then *conclude* that F has a number. So perhaps the proper background is a negative free logic in which identity statements, or perhaps atomic statements, entail existence, so that $a = b$ entails that a and b denote something.

Frege defines *zero* to be the number of the Fregean concept of being not self-identical, the concept characterized by the open formula $x \neq x$. The next item is the successor relation. Frege (1893, §76) writes:[12]

> I now propose to define the relation in which every two adjacent members of the series of [cardinal] numbers stand to each other. The proposition:
>
> > "there exists a concept G, and an object falling under it x, such that the Number which belongs to the concept G is n and the Number which belongs to the concept 'falling under G but not identical with x' is m"
>
> is to mean the same as
>
> > "n follows in the series of ... numbers directly after m".

Frege's definition, then, is that n follows directly after m just in case

$$\exists F \exists G((m = \#F \wedge n = \#G) \wedge \exists x(Gx \wedge \forall y(Fy \leftrightarrow (y \neq x \wedge Gy)))) \tag{Frege-successor}$$

The same definition is employed in Frege (1893) as well as in Wright (1983, p. 37).

The following is equivalent (via classical logic): the number n follows directly after the number m just in case:

$$\exists F \exists G\big(m = \#F \wedge n = \#G \wedge \exists x(\neg Fx \wedge \forall y(Gy \leftrightarrow (y = x \vee Fy)))\big) \tag{Successor}$$

To put it in Fregean words, n follows directly after m if and only if:

> There exists a concept F, and an object x not falling under F, such that the Number which belongs to the concept F is m and the Number which belongs to the concept 'either falling under F or identical with x' is n.

[12] The notation is tweaked slightly.

Let us write $\mathsf{S}(x,y)$ for "y bears (Successor) to x", or, in Fregean terms, "y follows directly after x".

In (Frege, 1893, §76), Frege immediately notes that he does not use the expression "*the* Number following next after m", since he had not yet shown that each number has exactly one successor. Of course, one can, and Frege does, prove uniqueness—using classical logic. Finally, Frege defines a *natural number* to be an ancestor of zero under (Frege-successor):

$$\mathbb{N}x \leftrightarrow_{\text{def}} \mathsf{S}^*(0,x)$$

With these definitions, the usual Dedekind-Peano axioms follow. Every natural number has a unique successor, and so the successor relation is a function; this function is one-to-one; zero has no successor, and full, second-order induction holds:

$$\forall X[(X0 \wedge \forall x \forall y((Xx \wedge \mathsf{S}(x,y)) \rightarrow Xy)) \rightarrow \forall x(\mathbb{N}x \rightarrow Xx)]$$

Induction is a straightforward consequence of the use of the ancestral in the definition of $\mathbb{N}$.

Where in the above, more or less standard development of this instance of abstractionism, do we run afoul of the Aristotelian dictum to reject any and all actual infinities, any completed infinite totalities? From the background actualism, we have that there are infinitely many natural numbers. That follows from the Dedekind-Peano principles established by Frege's Theorem(s). So the Fregean concept expressed by the formula $\mathsf{S}^*(0,x)$ holds of infinitely many things, all of the natural numbers. Moreover, an application of the '#' operator to $\mathbb{N}$ would give us the existence of an infinite number, one that Frege called "*endlos*". In modern terms, that is the cardinal number $\aleph_0$.

The ancestral relation is an essential part of the development and, in particular, the proof that the natural numbers satisfy the induction principle. Recall that for any binary relation R, R^*xy holds if *every* concept that holds of x and is closed under R also holds of y. But every concept that holds of 0 and is closed under S holds of infinitely many things. So, for our Aristotelian, there are no such concepts and so $\mathbb{N}$ vacuously holds of everything! This motivates the development of a potentialist, account of arithmetic, along Aristotelian and Fregean lines.[13]

4. Potentialist arithmetic

Linnebo (2013) and Chapter 3 of (Linnebo, 2018) presents a (consistent) account of set theory based on a potentialist version of Frege's Basic Law V. Instead of saying that every Fregean concept has an extension (which is inconsistent, given the usual comprehension principles of higher-order logic), we say that at least some Fregean concepts X *could* have an extension.

$$\Diamond \exists x(x = \varepsilon(X))$$

To invoke the heuristic of possible worlds, if all of the objects that X holds of are in a single world w, then there is an accessible world that contains an extension whose members are all and only the objects that X holds of in w.

To formulate this, Linnebo uses the language of plural logic. It is stipulated that, like sets, pluralities are *rigid*: the same objects are among a given plurality xx in all worlds in which xx exists.[14] As Linnebo (2013, 2018) shows, this rigidity can be expressed in the underlying modal and plural language.

The set-existence principle is

$$\Box \forall xx \Diamond \exists y \forall z(z \in y \leftrightarrow z \prec xx). \qquad \text{(set exists)}$$

[13] As is well-known, there are issues concerning how the underlying higher-order logic is to be developed in the framework. One issue is the analogue of impredicative comprehension. Sean Walsh (2012) (Corollary 92) shows that full first-order Dedekind-Peano arithmetic PA cannot be interpreted in predicative Frege arithmetic—regardless of our definitions of the arithmetical primitives. It follows from (Walsh, 2012), Corollary 92 and Proposition 6, that the interpretability strength of the system of second-order arithmetic known as ACA_0 is strictly above that of (HP) with Δ^1_1-comprehension, and hence also strictly above that of (HP) with predicative comprehension. These issues are not broached here.

[14] Clearly, if something is rigid then it is stable, but the converse typically fails.

In words, for any objects, there could be a set whose members are just those objects. On pain of Russell's paradox, there is no plurality of all possible sets, no plurality of all possible von Neumann ordinals, and the like.

We do not adopt (set exists) here since we are not (yet) interested in a potentialist set theory (but see §7 below). The present framework has both higher-order variables (ranging over Fregean concepts) and plural variables (ranging over rigid pluralities). Concerning Fregean concepts, the framework has full, unrestricted comprehension: each instance of the universal closure of the following holds:

$$\Box\exists R\Box\forall x_1\ldots\forall x_n(Rx_1\ldots x_n\leftrightarrow\Psi),$$

where Ψ is any formula that does not have R free. That is, every formula in the language defines a Fregean concept.[15]

Note that a Fregean concept can have different extensions in different worlds. For example, the concept of being self-identical (defined by "$x=x$") holds of different things in different worlds. The concept of being the largest number also holds of different numbers in different worlds. The same goes for a concept like being a living dog—dogs come and, alas, go. In short, unlike pluralities, concepts are not rigid.

There also is a comprehension principle for pluralities. Each instance of the universal closure of the following holds:

$$\Box(\exists x\Psi\rightarrow\exists xx\forall x(x\prec xx\leftrightarrow\Psi)),$$

where Ψ is a formula that does not contain xx free. That is, every formula in the language that holds of something defines a (rigid) plurality in each world. So if $\exists x\Psi$ holds in a world w, then the instance of plural comprehension defines a plurality of all objects that satisfy Ψ in w. Of course, those objects need not satisfy Ψ in a different world.

To avoid a free logic, we replace the abstractionist number-of operator with a relation. So for a Fregean concept F, '$\#(F,n)$' can be read as "n is a number of F". And similarly '$\#(mm,n)$' can be read as "n is a number of mm".[16]

As noted, for simplicity (and to keep the logic unfree), we also assume that objects are never destroyed—if an object exists in a given world, it exists in all accessible worlds. So the accessibility relation will be at least that of S4, and we adopt the above policy and take the accessibility relation to be convergent (if not linear).

4.1. Numbers of pluralities

In §2 we formulated and adopted a principle (Aristotle) that entails that, in effect, all worlds are Dedekind finite. It follows that all pluralities are Dedekind finite. We thus adapt (set exists) to numbers and adopt:

$$\Box\forall xx\Diamond\exists y\#(xx,y). \qquad \text{(Num Exists)}$$

In words, every plurality could have a number.

Define equinumerosity on pluralities as follows:

$$xx\approx yy\leftrightarrow_{\text{def}}\exists R(\forall x(x\prec xx\rightarrow\exists!y(Rxy\wedge y\prec yy))\wedge\forall y(y\prec yy\rightarrow\exists!x(Rxy\wedge x\prec xx))).$$

We do not have that every plurality has a number (see below for more details on this). The relevant version of Hume's Principle is that if two pluralities each have a number, then these numbers are identical just in case the pluralities are equinumerous:

$$\forall xx\forall yy\forall x\forall y((\#(xx,x)\wedge\#(yy,y))\rightarrow(x=y\leftrightarrow xx\approx yy)). \qquad \text{(HP)}$$

15 Since there are no restrictions on the formula Ψ, the system allows impredicative definitions. The same goes for plural comprehension. This is taken to be of-a-piece with a liberal view of potentialism. In any case the goal here is to sanction full induction on the natural numbers, and thus an analogue of full second-order arithmetic. Full impredicative comprehension facilitates that.

16 Some readers (and one referee) find it awkward or annoying or infelicitous to use the same symbol ('#') for both numbers of pluralities and numbers of Fregean concepts, even though context always settles which is meant. It is, of course, simple and straightforward to use two different symbols instead.

It follows that each plurality has at most one number:

$$\Box\forall xx\forall y\forall z((\#(xx,y)\wedge\#(xx,z))\rightarrow y=z).$$

We add an axiom that if two pluralities are equinumerous and one of them has a number, then so does the other:

$$\forall xx\forall yy\forall x((xx\approx yy\wedge\#(xx,x))\rightarrow\exists y\#(yy,y)). \qquad \text{(HP}'\text{)}$$

Of course, it follows from this and (HP) that these two numbers are identical: if two pluralities are equinumerous and one has a number, then the other has the same number.[17]

It seems reasonable to add a principle that if a certain number exists (in a given world) then so do all smaller numbers:

$$\Box\forall xx\forall yy\forall x((\#(xx,x)\wedge\forall z(z\prec yy\rightarrow z\prec xx))\rightarrow\exists w\#(yy,w)). \qquad \text{(Closure Down)}$$

The idea is that the numbers are generated one at a time, in their natural order.

One does not have to think of the numbers as generated this way. One might instead allow a given plurality to get a number before some of the its sub-pluralities have numbers. In that case, the relevant principle would be that if a plurality has a number, then it *could be* that all of its sub-pluralities have numbers:

$$\Box\forall xx\forall x(\#(xx,x)\rightarrow\Diamond\forall yy(\forall z(z\prec yy\rightarrow z\prec xx)\rightarrow\exists w\#(yy,w))). \qquad \text{(potential closure down)}$$

The principle (potential closure down) is a kind of generalization on the (G) axiom. From (G) and (Num Exists) we have that for any given finite list of pluralities in a given world, there is an accessible world that contains numbers for all of them. But, it seems, one cannot prove this generalization, at least as it is stated. Moreover, our (Aristotle) principle only entails that all worlds are Dedekind finite. And we certainly have no guarantee (yet) that the reasoning extends to Dedekind finite collections of pluralities.[18]

So far, we have the possibility of a world that contains only numbers, and which has a number for every plurality in that world. Consider, for example, a world whose domain is the first five numbers, starting with one. But we have such a world (or such a possibility) only because we have not yet introduced zero as a number. Following standard practice, there is no "empty" plurality, and so there is no plurality whose number is zero.

4.2. Numbers of Fregean concepts

One reason that we treat numbers of Fregean concepts here is to explicate Frege's account of how arithmetic is applied. Frege (1893, §§45-46) asks what is it that (cardinal) numbers are numbers of? What is it that we count? What do we apply number words to? His answer is that we apply numbers to concepts:

> While looking at one and the same external phenomenon, I can say with equal truth both "It is a copse" and "It is five trees", or both "Here are four companies" and "Here are 500 men". Now what changes here from one judgment to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is itself only a sign that one concept has been substituted for another. This suggests as the answer [to the question is] that the content of a statement of number is an assertion about a concept. This is perhaps clearest with the number 0. (§46)

This account is also adopted by the abstractionists.

We introduce zero as the number of a Fregean concept. One option is to add a constant "0" to the language, along with the axiom:

$$\Box\forall F(\#(F,0)\leftrightarrow\neg\exists xFx).$$

17 There is no need for an analogous axiom Linnebo's (2013, 2018) treatment of set theory. If two pluralities are coextensive, then they are the same plurality.

18 Thanks to Øystein Linnebo for these observations.

This entails that zero exists in every world or, in other words, zero necessarily exists. Since we are not presupposing a free logic, we have that every constant denotes something—the same thing—in every world.

To be sure, having numbers—any numbers—that exist of necessity is contrary to the spirit of potentialism. But perhaps the necessary existence of zero is at least relatively harmless: zero is the *only* number that exists necessarily.

We opt to avoid even this. Instead, we define a concept Z that, necessarily, holds of nothing. Following Frege, say that Z is defined by the formula $x \neq x$:

$$\Box(\forall x(\mathrm{Z}x \leftrightarrow x \neq x)).$$

Since we have comprehension, this concept constant is eliminable, but it is convenient to have it in the formal language. We add an axiom that it is necessary that zero could exist:

$$\Box\Diamond\exists x\#(\mathrm{Z},x). \qquad \text{(Zero)}$$

Following an informal Hume's Principle, we add an axiom stating that, necessarily, any number of Z is not the number of a plurality:

$$\Box\forall x\forall xx\neg(\#(Z,x)\wedge\#(xx,x)). \qquad \text{(Zero Not Plural)}$$

And we extend (Closure Down) to a statement that if any number exists, then zero does:[19]

$$\Box(\exists xx\exists y\#(xx,y)\rightarrow\exists x\#(\mathrm{Z},x)). \qquad \text{(Closure Down+)}$$

Let E be a concept constant defined by the formula "$x = x$". It is the concept of being self-identical. Like Z, this concept constant is eliminable. It is straightforward to see that E cannot have a number in any world that is actually finite. Suppose that a given world w has exactly n elements in its domain. If n exists in w then by (Closure Down) and (Closure Down+), every number smaller than n is also in w. There are thus $n+1$ numbers in W, which is a contradiction. Moreover, the plurality of all objects in this world w also has no number.

Consider an interpretation in which every world is actually finite. Then E cannot have a number in any world in that interpretation. So this interpretation satisfies:

$$\Box\neg\exists x\#(\mathrm{E},x),$$

or, in other words:

$$\Box\forall xx(\forall x(x \prec xx)\rightarrow\neg\exists x\#(xx,x)).$$

Recall our principle (Num Exists) stating that every plurality could have a number:

$$\Box\forall xx\Diamond\exists y\#(xx,y).$$

The concept-analogue of this would be:

$$\Box\forall X\Diamond\exists y\#(X,y).$$

As we have just seen, this is false at least in interpretations in which every world is actually finite. Our identity concept E is a counterexample to this principle in any such interpretation.

It might prove instructive to dwell on this. The problem that, unlike pluralities (as conceived here), at least some Fregean concepts are not rigid—and some are not stable. Let X be a concept, and focus on a particular world w. The above formula says that there is a world w', accessible from w that contains a number a number of X. What we would get, if the principle were correct, is the existence of a number of X *in* w', and *that* number may be different from the number of X in w. Let's apply this to our concept E. Suppose that a given world w

19 It is straightforward to modify (potential closure down) in an analogous way.

has n members. The above principle would tell us that there is a world w' in which E has a number. But this would be the number of elements in w', not the number of elements of w. Again, there is no such number in w'.

We add the following, connecting the numbers of non-empty Fregean concepts (in a given world) to pluralities in that world, the number of the objects that the concept holds of in that world:

$$\Box\forall X\forall xx\forall y(\#(xx,y)\wedge(\forall z(Xz\leftrightarrow z\prec xx))\rightarrow\#(X,y))$$

In words, and invoking the heuristic, if xx consists of the objects that X holds of in a given world, and if the number of xx is y, then the number of X is y in that world. Of course, the concept X may not have that same number (or any number) in a different world.

We add a converse:

$$\forall X\forall y(\#(X,y)\rightarrow\forall xx(\forall z(Xz\leftrightarrow z\prec xx)\rightarrow\#(xx,y)))$$

This says that if a given Fregean concept has a number (in a world) and a given plurality is co-extensive with the concept (in that world) then the plurality has the same number as the concept.

Define equinumerosity on Fregean concepts as follows:

$$F\approx G\leftrightarrow_{\text{def}}\exists R(\forall x(Fx\rightarrow\exists!y(Rxy\wedge Gy))\wedge\forall y(Gy\rightarrow\exists!x(Rxy\wedge Fx))).$$

then (HP), plus the usual properties of identity, entails:

$$\forall X\forall Y\forall x\forall y((\#(X,x)\wedge\#(Y,y))\rightarrow(x=y\leftrightarrow X\approx Y)).$$

In words, if two Fregean concepts have the same number (in a given world) then they are equinumerous (in that world). This, of course, is a concept version of (HP). This and (HP$'$) entails that if two concepts are equinumerous and one of them has a number, then the other one has the same number:

$$\forall X\forall Y\forall z(((X\approx Y\wedge\#(X,z))\rightarrow\forall w((\#(Y,w)))\leftrightarrow z=w)).$$

5. Frege's Theorem potentialized

5.1. The Dedekind-Peano axioms

A typical articulation of the (second-order) Dedekind-Peano axioms uses a language with a predicate N for being a natural number, a constant 0 for zero, and a successor function s. The axioms are that zero is a number, that the successor of a number is a number, that there is no number whose successor is zero, that the successor function is one to one on the numbers, and an axiom of induction—for any Fregean concept X, if X holds of zero and if X is closed under the successor function, then X holds of every number:

1. $\mathsf{N}(0)$
2. $\forall x(\mathsf{N}(x)\rightarrow\mathsf{N}(sx))$
3. $\neg\exists x(\mathsf{N}(x)\wedge sx=0)$
4. $\forall x\forall y((\mathsf{N}(x)\wedge\mathsf{N}(y)\wedge sx=sy)\rightarrow x=y)$
5. $\forall X((X0\wedge(\forall x((\mathsf{N}(x)\wedge Xx)\rightarrow Xsx)))\rightarrow(\forall x(\mathsf{N}(x)\rightarrow Xx)))$

Recall that, in the present modal potentialist setting, we avoid a free logic. Since we do not want any individual constants or any closed terms in the formal language, we follow Frege and employ a successor *relation*: $S(x,y)$ says that y is a successor of x. We use the following as our basic (non-modal) Dedekind-Peano axioms:

1. $\exists y(\mathsf{N}(y)\wedge\forall x\neg(\mathsf{N}(x)\wedge S(x,y)))$
2. $\forall x(\mathsf{N}(x)\rightarrow\exists y\forall z((\mathsf{N}(z)\wedge S(x,z))\leftrightarrow y=z))$
3. $\forall x\forall y\forall z((\mathsf{N}(x)\wedge\mathsf{N}(y)\wedge\mathsf{N}(z)\wedge S(x,z)\wedge S(y,z))\rightarrow x=y)$
4. $\forall X(\forall x((\mathsf{N}(x)\wedge\forall y\neg(\mathsf{N}(y)\wedge S(y,x))\rightarrow Xx)\wedge(\forall x((\mathsf{N}(x)\wedge Xx\wedge S(x,y))\rightarrow Xy)))\rightarrow(\forall x(\mathsf{N}(x)\rightarrow Xx)))$

The first axiom says that there is a number that is not a successor of anything; the second says that the successor relation is a function on numbers; the third says that this function is one to one; and the fourth is induction.

5.2. Definitions

Our next chore is to say what it is to be a natural number, in the potentialist setting, and to define the successor relation and some related items. Then we give potentialist versions of the Dedekind-Peano axioms, and establish those, along the lines of Frege's Theorem.

As noted, Frege defines a natural number to be an ancestor of zero under the successor relation. So, for Frege, a natural number is a finite cardinal number. Recall our axiom (Aristotle) designed to entail that, to invoke the heuristic, all worlds are Dedekind finite. There are no Dedekind infinite pluralities nor are there any Fregean concepts that apply to Dedekind infinitely many things (in any world). So here we take *all* (cardinal) numbers to be natural numbers. So define an object to be a *natural number* if it is the number of a Fregean concept:

$$\mathsf{NN}(x) \leftrightarrow_{\text{def}} \exists F \#(F,x)$$

It is straightforward to show that x is a natural number just in case either it is the number of the concept Z of being not self-identical or it is the number of a plurality:

$$\mathsf{NN}(x) \leftrightarrow (\#(\mathrm{Z},x) \vee \exists xx \#(xx,x)).$$

Let O be the concept of being the number of a plurality of just one thing:

$$\mathrm{O}(x) \leftrightarrow_{\text{def}} \exists xx((\#(xx,x) \wedge \forall y \forall z((y \prec xx \wedge z \prec xx) \rightarrow y = z)))$$

By (HP), if $\mathrm{O}(x)$ and $\mathrm{O}(y)$, then $x = y$. In effect, O is the concept of being the number one.

Frege and the abstractionist neo-logicists define the number one to be the number of the concept of being identical to zero. This can be captured here. Recall our concept Z of being not self identical. The following follows:

$$\forall x \forall y \forall xx((\#(\mathrm{Z},x) \wedge \forall z(z \prec xx \leftrightarrow z = x) \wedge \#(xx,y)) \rightarrow \mathrm{O}(y)).$$

In words, if y is the number of a plurality of only a number of Z, then O(y).

The next thing to be defined is the successor relation on numbers. Above, we gave a Fregean version of a successor relation, plus another one that is classically equivalent but perhaps more natural. Number n follows directly after number m just in case:

$$\exists F \exists G((m = \#F \wedge n = \#G) \wedge \exists x(Gx \wedge \forall y(Fy \leftrightarrow (y \neq x \wedge Gy)))) \qquad \text{(Frege-successor)}$$

$$\exists F \exists G\big(m = \#F \wedge n = \#G \wedge \exists x(\neg Fx \wedge \forall y(Gy \leftrightarrow (y = x \vee Fy)))\big) \qquad \text{(Successor)}$$

Those can be captured here. We use a version of the latter:

$$\mathsf{S}(m,n) \leftrightarrow_{\text{def}} \exists F \exists G(\#(F,m) \wedge \#(G,n) \wedge \exists x(\neg Fx \wedge \forall y(Gy \leftrightarrow (y = x \vee Fy))))$$

It can be shown that S is stable, since the concepts F and G can themselves be chosen to be stable, say as either the concept Z of being not self identical or the concept of being one of a given plurality. The following can be shown:

$$\begin{aligned}&\mathsf{S}(m,n) \leftrightarrow (\#(\mathrm{Z},m) \wedge \mathsf{O}(n)) \vee \\ &\exists xx \exists yy(\#(xx,m) \wedge \#(yy,n) \wedge \exists x(x \not\prec xx \wedge \forall y(y \prec yy \leftrightarrow (y = x \vee y \prec xx)))).\end{aligned}$$

In words, n is a successor of m just in case either m is a number of the concept of being not self identical and n is a number of a plurality of just one thing, or m is a number of a plurality xx and n is a number of a plurality of the xx and one more thing.

It will prove convenient to define the inequality relations on natural numbers:

$$x \leq y \leftrightarrow_{\text{def}} \exists F \exists G(\#(F,x) \wedge \#(G,x) \wedge \forall z(Fx \rightarrow Gx))$$

$$x < y \leftrightarrow_{\text{def}} (x \leq y \wedge x \neq y)$$

Again, we can see that these relations are stable since one can choose stable Fregean concepts F and G, either our empty concept Z or the concept of being one of a given plurality.

5.3. Targets

Recall that the potentialist translation of a non-modal formula is the result of replacing each universal quantifier $\forall$ with $\Box\forall$ and each existential quantifier $\exists$ with $\Diamond\exists$, and translating the connectives homophonically.

Our first three targets are the necessitations of the potentialist translations of the first three Dedekind-Peano axioms:

1. $\Box\Diamond\exists y(\mathsf{NN}(y) \wedge \Box\forall x\neg(\mathsf{NN}(x) \wedge \mathsf{S}(x,y)))$
2. $\Box\forall x(\mathsf{NN}(x) \rightarrow \Diamond\exists y\Box\forall z((\mathsf{NN}(z) \wedge \mathsf{S}(x,z)) \leftrightarrow y = z))$
3. $\Box\forall x\Box\forall y\Box\forall z((\mathsf{NN}(x) \wedge \mathsf{NN}(y) \wedge \mathsf{NN}(z) \wedge \mathsf{S}(x,z) \wedge \mathsf{S}(y,z)) \rightarrow x = y)$

We establish those here, and turn to induction in the next subsection. The first axiom is straightforward:

Theorem 1: $\Box\Diamond\exists y(\mathsf{NN}(y) \wedge \Box\forall x\neg(\mathsf{NN}(x) \wedge \mathsf{S}(x,y)))$

Proof: Recall our axiom (Zero):

$$\Box\Diamond\exists x\#(\mathrm{Z},x),$$

and our definition of the "empty" concept Z:

$$\Box(\forall x(\mathrm{Z}x \leftrightarrow x \neq x)).$$

It follows immediately from the definition of S that no number of Z can be a successor of another number.

We turn next to the second axiom. We break it up into two parts, first showing that successors are unique:

Theorem 2 (Frege): $\Box\forall x\forall y_1\forall y_2((\mathsf{S}(x,y_1) \wedge \mathsf{S}(x,y_2)) \rightarrow y_1 = y_2)$

Proof: Suppose that $\mathsf{S}(x,y_1)$ and $\mathsf{S}(x,y_2)$ both hold (in a given world). Then

$$\exists G_1\exists F_1(\#(G_1,x) \wedge \#(F_1,y_1) \wedge \exists w_1(\neg G_1 w_1 \wedge \forall w'(F_1 w' \leftrightarrow (w' = w_1 \vee G_1 w'))))$$

and

$$\exists G_2\exists F_2(\#(G_2,x) \wedge \#(F_2,y_2) \wedge \exists w_2(\neg G_2 w_2 \wedge \forall w'(F_2 w' \leftrightarrow (w' = w_2 \vee G_2 w'))))$$

We have to show that $y_1 = y_2$. Suppose that F_1, G_1, w_1, F_2, G_2, and w_2 have the relevant features. Then we have $\#(G_1,x)$ and $\#(G_2,x)$. By (HP), $G_1 \approx G_2$. So there is a relation R such that $\forall u(G_1 u \rightarrow \exists! v(Ruv \wedge G_2 v))$ and $\forall v(G_2 v \rightarrow \exists! u(Ruv \wedge G_1 u))$. Using comprehension, let $R'uv$ hold if and only either Ruv or else $u = w_1$ and $v = w_2$. It follows that $F_1 \approx F_2$. By (HP), $y_1 = y_2$.

We turn next to the third axiom, that if two numbers have a common successor, then they are identical:

Theorem 3 (Frege): $\Box\forall x\Box\forall y\Box\forall z((\mathsf{NN}(x) \wedge \mathsf{NN}(y) \wedge \mathsf{NN}(z) \wedge \mathsf{S}(x,z) \wedge \mathsf{S}(y,z)) \rightarrow x = y)$

Proof (sketch): Suppose that $\mathsf{S}(x,z)$ and $\mathsf{S}(y,z)$ both hold (in a given world). Then

$$\exists F_1\exists G_1(\#(F_1,x) \wedge \#(G_1,z) \wedge \exists w_1(\neg F_1 w_1 \wedge \forall w'(G_1 w' \leftrightarrow (w' = w_1 \vee F_1 w'))))$$

and

$$\exists F_2\exists G_2(\#(F_2,y) \wedge \#(G_2,z) \wedge \exists w_2(\neg F_2 w_2 \wedge \forall w'(G_2 w' \leftrightarrow (w' = w_2 \vee F_2 w'))))$$

We have to show that $x = y$. Suppose that F_1, G_1, w_1, F_2, G_2, and w_2 have the relevant features. Then we have $\#(G_1,z)$ and $\#(G_2,z)$. By (HP), $G_1 \approx G_2$. Our theorem follows from what may be called the *Equinumerosity Lemma*:

Lemma: Suppose that $X \approx Y$, Xx, and Yy. Let X' be the Fregean concept defined by $\forall z(X'z \leftrightarrow (Xz \wedge z \neq x))$, and let Y' be the Fregean concept defined by $\forall z(Y'z \leftrightarrow (Yz \wedge z \neq y))$. Then $X' \approx Y'$.

For a proof, see https://plato.stanford.edu/entries/frege-theorem/proof5.htm.

Notice that in the proofs of Theorems 1-3, we have not invoked the axiom (Aristotle) that all worlds are Dedekind finite (so to speak). We do so now.

Theorem 4: No plurality is equinumerous with a proper sub-plurality:

$$\Box\forall xx\forall yy((\forall x(x \prec xx \to x \prec yy) \land \exists y(y \prec yy \land y \not\prec xx)) \to xx \not\approx yy).$$

Proof: Suppose that $\forall xx\forall yy(\forall x(x \prec xx \to x \prec yy) \land y \prec yy \land y \not\prec xx)$. And suppose that $xx \approx yy$. Then there is a relation F that maps *xx* one-to-one onto *yy*, and a relation G that maps *yy* one-to-one onto *xx*. By first applying G and then F, we obtain a relation that maps *yy* one-to-one onto a proper subset of *yy*. This contradicts (Aristotle).

This result, together with (HP), sanctions (in this context) Euclid's Common Notion: a whole is greater than its (proper) parts.

Corollary: $\Box\forall n(\mathsf{NN}(n) \to \neg\mathsf{S}(n,n))$; no number is its own successor.

Proof: Suppose that $\mathsf{S}(x,x)$. That is,

$$\exists F\exists G(\#(F,n) \land \#(G,n) \land \exists x(\neg Fx \land \forall y(Gy \leftrightarrow (y = x \lor Fy))))$$

Let F, G, and x be given as in the formula:

$$\#(F,n) \land \#(G,n) \land \neg Fx \land \forall y(Gy \leftrightarrow (y = x \lor Fy)).$$

By (HP), $F \approx G$:

$$\exists R[\forall z(Fz \to \exists! y(Gw \land Rzw)) \land \forall w(Gw \to \exists! z(Xz \land Rzw))]$$

But since $\forall x(Gx \to Fx)$, R is a one-to-one function on G which is not a surjection: x is not in its range. This contradicts (Aristotle).[20]

Next is a common result in the abstractionist program. It does not depend on the (Aristotle) axiom.

Theorem 5: (Frege) Suppose that 0 is a (or the) number of our "empty" concept Z (in a given world w). Consider the concept N^F of being an ancestor of 0 (in w) under the successor relation:

$$\forall x(\mathsf{N}^F(x) \leftrightarrow \mathsf{S}^*(0,x))$$

Suppose $\mathsf{N}^F(n)$. Then n is either 0 or is the number of the plurality of all numbers (in w) that are less then n.

Proof: Let $A(x)$ be the Fregean concept of being either 0 or the number of numbers less than x. Of course $A(0)$. Suppose that $A(n)$ and that $\mathsf{S}(n,n')$, i.e., that n' is a successor of h and thus, from Theorem 2, n' is the successor of n. From the definition of the ancestral, we have to show that $A(n')$. We have that n is the number of numbers less than n, that $n \leq n'$, and, by the Corollary to Theorem 4, $n \neq n'$. By the definition of successor, n' is the number of numbers less than n'.

We now return to the other half of the second Dedekind-Peano axiom. We show that, necessarily, for every number n, it is possible that n has a successor:

[20] Suppose we are working in a system like ours but without the (Aristotle) axiom. Let *aa* be Dedekind infinite, and assume $\#(aa,n)$. Then it is straightforward to show that $\mathsf{S}(n,n)$; n *is* its own successor.

Theorem 6: (Frege) $\Box\forall x(\mathsf{NN}(x) \to \Diamond\exists y \mathsf{S}(x,y))$

Proof: We have that for every number x, either x is a number of Z (the Fregean concept of being not self identical), or there is a plurality xx such that x is a number of xx.

Suppose that 0 is a (or by (HP), the) number of Z. Consider the plurality aa of just 0. Recall our axiom (Num Exists):

$$\Box\forall xx\Diamond\exists y\#(xx,y).$$

This entails that aa could have a number. It is straightforward that this number is a successor of 0 (namely the number one).

So now let n be the number of a plurality aa (so that $\#(aa,n)$). To invoke the heuristic, suppose that aa exists in a world w. We have to show that n could have a successor.

Case 1: There is an object c in w such that $c \not\prec aa$. Consider the plurality aa' consisting of the aa and c. By (Num Exists), aa' could have a number. So there is a world w' accessible from w and a number n' of aa. It is straightforward that n' is a successor of n.

Case 2: There is no object c in w such that $c \not\prec aa$. In other words, aa is the plurality of all objects in w. We show that this is impossible. Recall our axioms (Closure Down) and (Closure Down+), that if a number exists in a world then so do all smaller numbers. So w contains the number n of our "universal" plurality aa and all smaller numbers. By (Closure Down+), it contains 0, the number of our "empty" concept Z. Let aa be any plurality of objects in w. Then any number of aa is less than our equal to n, So that number is also in w. So w contains the number of every plurality of objects in w.

Let nn be the plurality of all ancestors of 0 in w. That is

$$\forall x(x \prec nn \leftrightarrow \mathsf{S}^*(0,x)).$$

Now if every nn had a successor in nn, then the successor relation S would be a one-to-one function on nn that is not a surjection (since 0 has no predecessor). This contradicts (Aristotle).

So there is a number m in nn that has no successor in nn. By Theorem 5, m is the number of all numbers less than m. But nn is a plurality of objects in w and so the number m' of nn is in the world w. But m' is the successor of m, and so m has a successor in w after all. This, of course, is a contradiction.

Theorem 7: $\Box\forall x((\mathsf{NN}(x) \wedge \neg\exists y\mathsf{S}(y,x)) \to \#(\mathrm{Z},x))$. The only number that does not have a predecessor is zero.

Proof: Suppose not, that

$$\Diamond\exists x(\mathsf{NN}(x) \wedge \neg\exists y\mathsf{S}(y,x) \wedge \neg\#(\mathrm{Z},x)).$$

Invoking the heuristic, there is a world w which contains a number a which is not a number of Z and which as no predecessor (in w). It follows that a is the number of a plurality (in w): $\exists xx\#(xx,a)$. Let aa be one such plurality (in w). So $\#(aa,a)$.

Recall our axioms (Closure Down) and (Closure Down+) that if a number exists (in a world) then so do all smaller numbers. Let A be the Fregean concept of being a number (in w) that is strictly smaller than a:

$$\forall z(Az \leftrightarrow (\#(\mathrm{Z},z) \vee \exists zz(\#(zz,z) \wedge \forall y((y \prec zz \to y \prec aa) \wedge zz \not\approx aa))))$$

Lemma: A is closed under successor (in w): $\forall u(A(u) \to \exists v(\mathsf{S}(u,v) \wedge A(v)))$.
Proof: Suppose $A(e)$. So either e is a number of Z or e is the number of a plurality ee that maps one to one, but not onto aa.

Case 1: $\#(\mathrm{Z},e)$, or, in words, e is a number of Z. Since pluralities are not "empty", let $c \prec aa$. Then consider the plurality cc of just c: $\forall x(x \prec cc \leftrightarrow x = c)$. Clearly, this plurality maps one to one into aa. Moreover, there is another element $d \neq c$ such that $d \prec aa$. Otherwise a would have a predecessor, namely e). So cc has a number (in w). That number is a successor of e.

Case 2: e is not a number of Z. Then there is a plurality ee such that $\#(ee,e)$ and ee is a sub-plurality of aa, but is not all of aa:

$$\forall(x \prec ee \rightarrow x \prec aa),$$

$$\exists x(x \prec aa \wedge x \not\prec ee).$$

So let $d \prec aa$ and $d \not\prec ee$. Then consider the plurality gg where $\forall x(x \prec gg \leftrightarrow (x \prec ee \vee x = d))$. Then, by (Closure Down) we have gg has a number g (in w) and so $\mathsf{S}(e,g)$. Moreover, gg is not all of aa. If it were, then we would have $\mathsf{S}(g,a)$, and so a would have a predecessor. So $A(g)$.

So we have that A holds only of numbers, and each such number has a successor that A holds of. And successors are unique. So, by (Aristotle), it follows that every member of A has a predecessor. But we also have that A holds of a number of Z, and we know that no such number can have a predecessor.

5.4. Induction

We turn to induction. Recall that Frege defines a natural number to be an ancestor of zero under the successor relation. There is what we may call a local version of induction:[21]

Theorem 8: $\Box\forall x(\mathsf{NN}(x) \leftrightarrow \exists z(\#(\mathrm{Z},z) \wedge \mathsf{S}^*(z,x))$

In words, x is a number (in a world w) just in case x is an ancestor of a number of Z under S.

Proof: Suppose that our theorem is false. To invoke our heuristic, suppose that the theorem fails in world w. Let a be an object in w. Suppose $\exists z(\#((\mathrm{Z},z) \wedge \mathsf{S}^*(z,a)))$. Then $\mathsf{NN}(x)$, since the relata of S are all numbers. So the right-to-left direction of the biconditional holds in w (trivially). So in w, there is a number 0 of Z and a number n which is not an ancestor of 0 under S. By our axioms (Closure Down) and (Closure Down+), every number less than or equal to n exists in w.

Let A be the Fregean concept of being a number less than our equal to n which is not an ancestor of 0 under S:

$$\forall x(A(x) \leftrightarrow (x \leq n \wedge \neg\mathsf{S}^*(0,n))).$$

Notice that $\neg A(0)$ since, of course, $\mathsf{S}^*(0,0)$. So if $A(m)$ then $m \neq 0$. So, by Theorem 8, m has a predecessor m'. And we have $A(m')$, since $m' < m \leq n$ and m' is not an ancestor of 0 under S (since otherwise m would be).

Let nn be the plurality of all objects of which A holds. We have that the predecessor relation is a one-to-one function on nn. So, by (Aristotle) the predecessor relation is onto nn. So n has a *successor* n' such that $A(n')$. But this is absurd, since n' is not less than or equal to n.

Recall our principle (finite) that states that, in effect, all worlds are finite:

$$\Box\exists R(\forall x\forall y_1\forall y_2((Rxy_1 \wedge Rxy_2) \rightarrow y_1 = y_2) \wedge \exists x\forall zR^*(x,z) \wedge \exists y\forall z(\neg R(y,z))) \qquad \text{(finite)}$$

We did not adopt (finite) as an axiom, opting for the ostensibly weaker (Aristotle), which only states that, in effect, all worlds are Dedekind finite.

We can now prove that all worlds are actually finite.

[21] Thanks to Øystein Linnebo for suggesting this.

Corollary: (finite)

Proof: Suppose w is any world that is Dedekind finite but not finite, and let aa be the plurality of being a object in w. Let n be a number of aa, presumably in a different, accessible world. By Theorem 8, n is an ancestor of zero under s.

So we see that (Closure Down) is a powerful axiom, as is (potential closure down). If we have a number n of a Dedekind finite, but not finite Fregean concept A (or plurality aa), in a given world, then by (Closure Down) (or (potential closure down)), there is (or could be) a Dedekind infinite sequence of numbers below n, contradicting (Aristotle). In short, all worlds are actually finite.[22]

Recall the formulation of induction in a non-modal language that has no constants and successor is chararized as a relation:

$$\forall X(\forall x((\mathsf{N}(x) \wedge \forall y \neg(\mathsf{N}(y) \wedge S(x,y)) \rightarrow Xx) \wedge (\forall x \forall y((\mathsf{N}(x) \wedge Xx \wedge S(y,x)) \rightarrow Xy))) \rightarrow (\forall x(\mathsf{N}(x) \rightarrow Xx))).$$

We can simplify this a little, using present notation:

$$\forall X(((\forall x\#(\mathsf{Z},x) \rightarrow Xx) \wedge (\forall x \forall y((\mathsf{NN}(x) \wedge Xx \wedge \mathsf{S}(y,x)) \rightarrow Xy))) \rightarrow (\forall x(\mathsf{NN}(x) \rightarrow Xx))).$$

The potentialist translation of this is:

$$\Box\forall X(((\Box\forall x(\#(\mathsf{Z},x) \rightarrow Xx) \wedge (\Box\forall x \forall y((\mathsf{NN}(x) \wedge Xx \wedge \mathsf{S}(y,x)) \rightarrow Xy))) \rightarrow (\Box\forall x(\mathsf{NN}(x) \rightarrow Xx))). \qquad (\text{Ind}^{\Diamond})$$

In words, and using the heuristic of possible worlds, (Ind$^{\Diamond}$) says that IF in any world w_1, if w_1 contains a number of Z, then X holds of that number (in w_1), and IF for any world w_2, if X holds of a number n in w_2 and a successor of n exists in w_2, then X holds of that successor (in $w2$), THEN for any world w_3, X holds of all of the numbers in w_3. The consequent says that whenever a number is generated, X necessarily holds of it.

Theorem 9: (Ind$^{\Diamond}$)

Proof: Suppose not. So there is world w and a Fregean concept A such that necessarily, if there is number of Z then A holds of that number and, necessarily, if A holds of a number n in w and that number has a successor in w, then A holds of that successor in w, and there is a number m in w and A does not hold of m in w.

By (Closure Down) and (Closure Down+), w contains all numbers smaller than m. Define a Fregean concept B as follows:

$$\forall x(Bx \leftrightarrow (x \leq m \wedge \neg Ax))$$

Of course, we have Bm and if n is a number of Z (i.e., $\#(\mathrm{Z},n)$), then $\neg Bn$, since A necessarily holds of any number of Z. Suppose that Bq and that q is predecessor of r, so that $\mathsf{S}(q,r)$. Then Br, since otherwise we would have Aq. So every number that B applies to has a predecessor that B applies to. By (Aristotle), m must have a successor that B applies to. But, by definition, everything that has B is less than or equal to m. This is a contradiction.

[22] The usual proof, in set theory, that if a set is Dedekind finite then it is finite uses the axiom of choice. Various choice principles can be formulated in pure higher-order logic (see Siskind, Mancosu and Shapiro, 2023). Our (Aristotle) and (finite) can be shown equivalent in a third-order language that has a sort for relations between pluralities and objects, and a sort for functions from pluralities to objects, plus the following choice principle:

$$\forall R(\forall xx \exists y R(xx,y) \rightarrow \exists F \forall xx Rxx, F(xx)).$$

5.5. Mirroring?

Recall the potentialist translation of our non-modal language: replace all quantifiers of the form $\forall x$ and $\forall X$ with $\Box\forall x$ and $\Box\forall X$ respectively, and replace all quantifiers of the form $\exists x$ and $\exists X$ with $\Diamond\forall x$ and $\Diamond\forall X$ respectively. If φ is a formula in the non-modal language, let $\varphi^{\Diamond}$ be its translation.

Recall also the classical potentialist mirroring theorem (from Linnebo, 2013 and 2018): Let $\vdash$ be the relation of classical deducibility in a non-modal *first-order* language $\mathscr{L}$. Let $\mathscr{L}^{\Diamond}$ be the corresponding modal language, and let $\vdash^{\Diamond}$ be deducibility in this language corresponding by $\vdash$, S4.2, and axioms asserting the stability of all atomic predicates of $\mathscr{L}$. Then for any formulas $\varphi_1,\ldots,\varphi_n,\psi$ of $\mathscr{L}$, we have:

$$\varphi_1,\ldots,\varphi_n \vdash \psi \quad \text{if and only if} \quad \varphi_1^{\Diamond},\ldots,\varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

The present non-modal language is, of course, second-order. For present purposes, we consider it to be a multi-sorted first-order language (with one sort for each kind of relation variable). There are two options for the target modal theory. One is to take the potentialist translations of the comprehension axioms as part of modal system $\vdash^{\Diamond}$, and the other is to include any use of comprehension as among the (non-logical) premises (the φ_i's) on the left hand side of the biconditional. Each of these follows in our modal theory (noting Lemma 5.3 of Linnebo, 2013).

Unfortunately, we cannot make direct use of the classical potentialist mirroring theorem here, for at least two reasons. First, the mirroring theorem requires all atomic formulas (in the modal language) to be stable, but our predication relation (between, say, monadic Fregean concepts and individual objects) is not stable. Consider the Fregean concept of being the largest number (in a given world):

$$Lx \leftrightarrow (\mathsf{NN}(x) \land \forall y(\mathsf{NN}(y) \to y \leq x))$$

Consider a world w that has a number n for Z, and no other numbers. Then Ln holds at w but not at any accessible world that contains more numbers. A second example is the concept of being the number of numbers (in a given world): $\#(\mathsf{NN},x)$. A world with, say, exactly four numbers has access to a world with exactly five. A second reason why the mirroring theorem does not (directly) apply is that the present modal theory has axioms (such as (Aristotle)) that are not in the range of the potentialist translation.[23]

We can, however, make some use of the mirroring theorem. Let PA$^{\Diamond}$ be the modal theory consisting of S4.2, axioms stating the stability of all atomic predicates of the above Dedekind-Peano theory, including the stability of predication, as well as the potentialist translations of the instances of the comprehension scheme of the non-modal theory. Note that we do not have full comprehension in PA$^{\Diamond}$. Let $\vdash^{\Diamond}$ be deducibility in PA$^{\Diamond}$.

The mirroring theorem thus applies to PA$^{\Diamond}$, with respect to our non-modal Dedekind-Peano arithmetic: For any formulas $\varphi_1,\ldots,\varphi_n,\psi$ of the non-modal Dedekind-Peano arithmetic, we have:

$$\varphi_1,\ldots,\varphi_n \vdash \psi \text{ in the non-modal theory if and only if } \quad \varphi_1^{\Diamond},\ldots,\varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

It is possible to interpret PA$^{\Diamond}$ in our present modal (and plural) theory (which does have full modal and plural comprehension, along with axioms that are not in the range of the potentialist translation). First, if X is a concept variable, then let SX be an abbreviation of the statement that X is stable:

$$SX \leftrightarrow_{\text{def}} (\forall x(Xx \to \Box Xx) \land \forall x(\neg Xx \to \Box\neg Xx))$$

and similarly for relation variables. Let ϕ be any formula of our modal and plural theory, and let ϕ^S be the result of restricting all higher-order variables in ϕ to S. That is, replace each $\forall X\psi$ with $\forall X(SX \to \psi)$ and replace each $\exists X\psi$ with $\exists X(SX \land \psi)$. It is straightforward (but tedious) to verify that if

$$\varphi_1^{\Diamond},\ldots,\varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond},$$

then $\psi^{\Diamond S}$ can be derived from $\varphi_1^{\Diamond S},\ldots,\varphi_n^{\Diamond S}$ in the present modal system.

We can do a lot better, however, since we only need one direction of the mirroring theorem.

23 There also may be an issue with the fact that the modal language has plural terminology, but that does not seem to be problematic.

Theorem 10: let $\vdash$ be deducibility in our original non-modal second-order theory of arithmetic, and let $\vdash^P$ be deducibility in our modal theory above. Then for any formulas $\varphi_1, \ldots, \varphi_n, \psi$ of the non-modal language:

$$\text{if} \quad \varphi_1, \ldots, \varphi_n \vdash \psi \quad \text{then} \quad \varphi_1^\Diamond, \ldots, \varphi_n^\Diamond \vdash^P \psi^\Diamond.$$

Proof: This is a straightforward (if tedious) induction on the length of a derivation in the non-modal theory of arithmetic (as in the proof of the mirroring theorem in Linnebo, 2013 and 2018).

Recall that in our modal theory, we have the potentialist translations of the instances of comprehension as well as the potentialist translations of all of the Dedekind-Peano axioms. So we also have potentialist translations of every theorem of second-order Dedekind-Peano arithmetic. Putting aside the obvious anachronism, we thus have a verification of an arithmetic counterpart to one of Aristotle's claim's about actual infinity. To repeat:

> Our account does not rob the mathematicians of their study, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the <actual> infinite . . . (207b27-30)

Of course, this applies to the mathematics of Aristotle's day, not the contemporary scene. Present focus is only on contemporary Dedekind-Peano arithmetic.

For the record, note that the following is a model for our modal theory: for each natural number n, there is a world whose domain is $\{m|m \leq n\}$, and accessibility is inclusion. So our modal theory is consistent if arithmetic is.

6. Doing without (Aristotle) and doing without (finite)

Recall our axiom (Aristotle) stating (in effect) that all worlds are Dedekind finite:

$$\Box\forall R\forall X[\forall x(((Xx \to \exists y\forall z(Xz \wedge Rxz) \leftrightarrow y = z)) \wedge \forall x_1\forall x_2\forall y((Rx_1y \wedge Rx_2y) \to x_1 = x_2)) \\ \to \forall y(Xy \to \exists x(Xx \wedge Rxy)))].$$

The plan here is to drop this. We do not want to *assert*, in the theory itself, the possible existence of, say, a plurality of all natural numbers, but we also don't want to rule out the possible existence of such a plurality. So we do not assert (here) the negation of (Aristotle), which would entail the existence of a world with a Dedekind infinite domain. Of course, we also do not wish to assert, here, the stronger principle (finite), that all worlds are actually finite, nor do we assert its negation.

Recall that, assuming (Aristotle) we defined a natural number to be the number of a Fregean concept or, equivalently, either a number of Z or the number of a plurality:

$$\mathsf{NN}(x) \leftrightarrow_{\text{def}} \exists F\#(F,x),$$

$$\mathsf{NN}(x) \leftrightarrow (\#(\mathsf{Z},x) \vee \exists xx\#(xx,x))$$

We do not want to say that here, of course. Instead, we take the formula(s) just above to be a definition of *number*:

$$\mathsf{N}(x) \leftrightarrow_{\text{def}} \exists F\#(F,x),$$

or, equivalently:

$$\mathsf{N}(x) \leftrightarrow (\#(\mathsf{Z},x) \vee \exists xx\#(xx,x)).$$

Now we need a definition of *natural* number, noting that there may be numbers that are not natural numbers. Recall that Frege's definition of the ancestral can be invoked in the potentialist context, provided it is formulated in terms of relations instead of functions, and does not invoke any singular terms. To deploy the heuristic, the definition works as advertised *within* each world. To remind the gentle reader, let R be a two place relation:

$$R^*xy \leftrightarrow_{\text{def}} \forall X[(Xx \wedge (\forall z \forall w((Xz \wedge Rzw) \rightarrow Xw))) \rightarrow Xy]$$

Recall also that Theorem 8 is what might be called a local version of induction:

$$\Box\forall x(\mathsf{NN}(x) \leftrightarrow \exists z(\#((\mathrm{Z},z) \wedge \mathsf{S}^*(z,x))))$$

This assumed our prior definition of a natural number as the number of any plurality or Fregean concept, and the proof invoked our (Aristotle) axiom. Here we follow Frege and just take this formula to be a definition of natural number:

$$\mathsf{NN}(x) \leftrightarrow_{\text{def}} \exists z(\#((\mathrm{Z},z) \wedge \mathsf{S}^*(z,x))) \qquad \text{(Frege)}$$

It is straightforward to verify that NN is stable, and that all natural numbers are numbers:

$$\Box\forall n(\mathsf{NN}(n) \rightarrow \mathsf{N}(x))$$

Recall our result that each ancestor of zero under successor is the number of all smaller numbers:

> **Theorem 5:** Suppose that 0 is a (or the) number of our "empty" concept Z (in a given world w). Consider the concept N^F of being an ancestor of 0 (in w) under the successor relation:
>
> $$\forall x(\mathsf{N}^F(x) \leftrightarrow \mathsf{S}^*(0,x))$$
>
> Suppose $\mathsf{N}^F(n)$. Then n is either 0 or is the number of the plurality of all numbers (in w) that are less then n.

Since this is (or could have been) established without invoking (Aristotle), it holds here: every natural number is the number of all smaller numbers.

In the above treatment (assuming (Aristotle)), our first three targets were the necessitations of the potentialist translations of the first three Dedekind-Peano axioms:

1. $\Box\Diamond\exists x(\mathsf{NN}(x) \wedge \forall y \neg(\mathsf{NN}(y) \wedge \mathsf{S}(x,y)))$
2. $\Box\forall x(\mathsf{NN}(x) \rightarrow \Diamond\exists y \Box\forall z((\mathsf{NN}(z) \wedge \mathsf{S}(x,z)) \leftrightarrow y = z))$
3. $\Box\forall x \Box\forall y \Box\forall z((\mathsf{NN}(x) \wedge \mathsf{NN}(y) \wedge \mathsf{NN}(z) \wedge \mathsf{S}(z,x) \wedge \mathsf{S}(z,y)) \rightarrow x = y)$

In words, (1) there is a number that is not a successor of anything, (2) every number has a unique successor (i.e., the successor relation is a function), (3) successor is one to one.

In the previous section, Theorem 1 established the first of these, Theorem 2 established "half" of the second, that successors are one to one, and Theorem 3 established the third. The proofs of these did not invoke the (Aristotle) axiom, and they relied on the definition of a natural number as the number of a Fregean concept. So they hold here for *numbers*, and not just natural numbers. We have:

1. $\Box\Diamond\exists x(\mathsf{N}(x) \wedge \forall y \neg(\mathsf{N}(y) \wedge \mathsf{S}(x,y)))$
2. $\Box\forall x \forall y_1 \forall y_2((\mathsf{S}(x,y_1) \wedge \mathsf{S}(x,y_2)) \rightarrow y_1 = y_2)$
3. $\Box\forall x \Box\forall y \Box\forall z((\mathsf{N}(x) \wedge \mathsf{N}(y) \wedge \mathsf{N}(z) \wedge \mathsf{S}(z,x) \wedge \mathsf{S}(z,y)) \rightarrow x = y)$

A fortiori, since all natural numbers are numbers, these hold for natural numbers as well.

Without (Aristotle) and (finite), we would not expect Theorem 4 above, that no plurality is equinumerous with a proper sub-plurality to hold. Consider the Corollary to Theorem 4:

$\Box\forall n(\mathsf{NN}(n) \to \neg\mathsf{S}(n,n))$; no number is its own successor.

This, too, relied on (Aristotle).

As indicated by the gloss, there are two statements to be pondered here. The first, taking the gloss literally (and out of its original context), is that no *number* is its own successor. Of course, we should not expect that to hold here, since we are not ruling out worlds with infinitely many members. As pointed out in note 20, if an interpretation has a Dedekind-infinite plurality *aa* in a given world, then the number of *aa* will be its own successor.

The other reading of Theorem 5 is that no *natural number* is its own successor. That follows from (Frege) and the definition of the ancestral:

Theorem 11: $\Box\forall x(\mathsf{NN}(x) \to \neg\mathsf{S}(x,x))$.

Proof: Let B be the Fregean concept which holds of something just in case it is a natural number that is not its own successor:

$$B(x) \leftrightarrow (\mathsf{NN}(n) \wedge \neg\mathsf{S}(n,n))$$

Let 0 be a (or, better, the) number of Z (the Fregean concept of being not self-identical). Since 0 has no predecessor, it is not a predecessor of itself, and so it is not a successor of itself. So $B(0)$. Now suppose that $B(x)$ and $\mathsf{S}(x,y)$. Then $\mathsf{NN}(y)$ and y has a predecessor, namely x. So $B(y)$. So we have $\mathsf{S}^*(a,y)$ So by (Frege) $\mathsf{NN}(y)$. Thus, B holds of all natural numbers.

Now we can easily establish the other "half" of the second Dedekind-Peano axiom, that every natural number could have a successor:

Theorem 12: (Frege) $\Box\forall x(\mathsf{NN}(x) \to \Diamond\exists y\mathsf{S}(x,y))$.

Proof: Suppose $\mathsf{NN}(n)$. By closure down, all numbers smaller then n exist. By Theorem 5, n is the number of the plurality of all numbers smaller than n. Let nn be the plurality of n and all numbers smaller than n. By (Num Exists) nn could have a number. This number is a successor of n.

The final axiom is induction. Recall that the potentialist translation of this is:

$$\Box\forall X(((\Box\forall x(\#(\mathsf{Z},x) \to Xx) \wedge (\Box\forall x\forall y((\mathsf{NN}(x) \wedge Xx \wedge \mathsf{S}(x,y)) \to Xy))) \to (\Box\forall x(\mathsf{NN}(x) \to Xx))). \qquad (\text{Ind}^{\Diamond})$$

Here the proof of this is straightforward.

Theorem 13: ($\text{Ind}^{\Diamond}$)

Proof: Let X be a Fregean concept and suppose that, necessarily, X holds of every number of Z, and that, necessarily, X is closed under successors. Let w be a world. Then if w contains a number of Z, then X holds of that number, and we have that X is closed under successor in w. So, by (Frege), X holds of all numbers in w.

So, as with the previous theory based on (Aristotle), the present theory has the potentialist translations of all theorems of second-order Dedekind-Peano arithmetic.

Recall that most of the proofs in the previous section that relied on (Aristotle) used a reductio, sometimes called classical reductio or negation introduction. Here all of the relevant proofs are constructive. So if the background logic is intuitionistic, the theory proves all theorems of full second-order Heyting arithmetic.

Finally, here is a model of the present theory, one that does satisfy the negation of (Aristotle) and thus the negation of (finite). There is one world that contains every natural number and also $\aleph_0$ (i.e., the number that Frege calls *endlos*), and for each natural number n, there is a world which contains all numbers less than or equal to n. Accessibility is inclusion.

7. Aristotelian set theory

The final project here is an Aristotelian set theory. Its intended interpretation is the hereditarily finite sets. As noted, Linnebo (2013, 2018) develop a potentialist set theory, based in a version of Frege's Basic Law V. The main idea is that, necessarily, for every plurality *aa*, there could be set whose members are the *aa*. There are axioms that make the theory equi-consistent with ZFC. The analogue of the axiom of infinity is an axiom stating that there is an transfinite stage or world, one that contains the set of all finite von Neumann ordinals, for example.

A natural attempt here would be a theory like Linnebo's, but with the aforementioned "infinity" axiom replaced with its negation, perhaps a set-theoretic analogue of our (Aristotle) axiom. Unfortunately, this will not do. It is "folklore" that ZFC^-—ZFC with the axiom of infinity replaced by its negation—is equivalent, in some sense, to Dedekind-Peano arithmetic. The fact is that the theories are mutually interpretable, but they are not definitionally equivalent. Moreover, Kaye and Wong (2007) point out that ZFC^- does not prove induction for membership, nor does it prove that every set has a transitive closure. Indeed, ZFC^- does not prove that every set is a subset of a transitive set, a principle sometimes called "transitive containment". ZFC^- is a rather bizarre and unnatural theory.

Because of the Mirroring theorem, the same goes for a potentialist set theory like the envisioned one by Linnebo, but with the "infinity" axiom replaced by its negation. The plan here is to start with the non-modal "Small Set Theory" (SST) of McCarty, Shapiro, and Rathjen (2024). The only non-logical symbol is that for membership. There are four axioms:

1. **Extensionality:** $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.
2. **Empty Set:** $\exists x \forall y . y \notin x$.

   We use "0" as a symbol for the empty set.
3. **Adjunction** $\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u \in x \vee u = y))$.

   Our unofficial (and eliminable) notation the adjunction of x and y is

   $$x \cup \{y\}.$$

   Note that, when writing $x \cup \{y\}$ here we do not presume thereby that an operation of binary union exists over the class of all sets. This is a theorem.
4. **Induction on Adjunction:** For any formula $\phi(x)$ in the language of set theory—featuring perhaps set parameters—if $\phi(0)$ and if

   $$\forall x \forall y ((y \notin x \wedge \phi(x) \wedge \phi(y)) \rightarrow \phi(x \cup \{y\})),$$

   then $\forall x \phi(x)$.

The background logic for this theory is intuitionistic. All of the axioms of ZFC, except, of course, Infinity, follow from these axioms, as well as induction for membership, a theorem that every set has a transitive closure, and a theorem that every set is finite. SST is definitionally equivalent to Heyting arithmetic. If the background logic is classical, then the theory is definitionally equivalent to first-order Dedekind-Peano arithmetic.

Note that SST is first-order, with no variables or symbols for either pluralities or Fregean concepts. The plan is to develop a potentialist version $SST^{\Diamond}$ of SST. We start with axioms assuring that sets are rigid, that they have the same members in all worlds. We add the potentialist translations of the four axioms of SST. Extensionality is:

$$\Box \forall x \forall y (\Box \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

Empty Set is

$$\Diamond \exists x \Box \forall y . y \notin x.$$

As with the Aristotelian arithmetic(s), we do not introduce any individual constants, since we do not wish to presuppose that any particular things exist. We introduce $\mathrm{E}(x)$ as an abbreviation of $\Box\forall y. y \notin x$. So our axiom is $\Diamond\exists x\mathrm{E}(x)$. Adjunction is

$$\Box\forall x\forall y\Diamond\exists z\Box\forall u(u \in z \leftrightarrow (u \in x \vee u = y)).$$

We introduce $\mathrm{A}(z,x,y)$ for z is an adjunction of x and y: $\Box\forall u(u \in z \leftrightarrow (u \in x \vee u = y))$.

Finally, induction on Adjunction: Necessarily, for any formula $\phi(x)$ in the language of the modal set theory featuring perhaps set parameters) if $\Box\forall x(\mathrm{E}(x) \rightarrow \phi(x))$ and if

$$\Box\forall x\forall y\forall z((y \notin x \wedge \phi(x) \wedge \phi(y) \wedge \mathrm{A}(z,x,y)) \rightarrow \phi(z))),$$

then $\Box\forall x\phi(x)$.

That's it. Since the language is first-order and membership is stable (indeed rigid), then the mirroring theorem entails that the potentialist translations of all of the axioms of ZFC, except, of course, Infinity, follow from these axioms, as well as induction for membership. We have that for every set x, there could be a transitive closure of x, and, crucially, it is necessary that every set is finite, and thus that every set is hereditarily finite. Our theory $\mathrm{SST}^{\Diamond}$ is definitionally equivalent to modalized Dedekind-Peano arithmetic.[24]

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[24] A different potentialist theory of the hereditarily finite sets can be obtained from Button (2021a, 2021b, 2021c) by adding axioms to the effect that every set could have a singleton and that there cannot be an infinite set.

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