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Against Plural Comprehension

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Abstract: Plural primitivism is the idea that plural expressions cannot be dispensed with in favor of singular expressions. Our current standard first-order logic is based on the opposite idea, *singularism*, that plural expressions are eliminable in terms of singular expressions. Hence, plural primitivism suggests replacing first-order logic with what is nowadays called *plural logic*. One prominent axiom of plural logic is the axiom scheme of *plural comprehension* (PCA). This article aims to critically examine the plural primitivist claim of the logicality of PCA.

Keywords: Plural Logic, Plural Comprehension, Second-order Logic, Plurals.

1. Introduction

What I call *plural primitivism* is the idea that plural expressions cannot be dispensed with in favor of singular expressions and should be counted in the irreducible primitive vocabulary of our logic. Our current standard first-order logic is based on the opposite idea, *singularism*, that plural expressions are eliminable in terms of singular expressions: first-order logic only admits singular terms and singular predications about individual objects. Hence, plural primitivism suggests replacing first-order logic with what is nowadays called *plural logic*.[1]

The syntax of plural logic augments that of first-order logic with additional syntactic categories of *plural terms*, including *plural variables* which we will denote by xx, yy, and zz, and *plural quantifiers*, which we will denote by $\forall xx$, $\forall yy$, and $\forall zz$. In addition to them, plural logic has a special binary logical predicate $\prec$ that takes a singular term t in the first argument place and a plural term tt in the second and thereby expresses 't is one of tt'.[2]

[1] This article follows the formalism of plural logic in (Rayo, 2002) and (Florio and Linnebo, 2021), which is called *plural first-order logic* PFO. There are other formalisms that employ different notations and choices of logical vocabulary, but my arguments in this article do not depend on how plural logic is formalized; see (Oliver and Smiley, 2016, Ch. 7) for a comprehensive list of different formalisms of plural logic in the literature.

[2] We may add *non-logical* plural predicates to PFO, which take plural terms (possibly as well as singular terms) as their arguments, and the resulting system is called PFO$^+$ in (Florio and Linnebo, 2021; Rayo, 2002). This article is primarily concerned with plural terms and quantifiers and does not discuss PFO$^+$, but all the criticisms of PFO I will present below equally apply to PFO$^+$.

How is this new formal glossary to be understood? One of the central tenets of plural primitivism is the *ontological innocence* of plural terms and quantifiers. According to plural primitivism, plural terms and quantifiers have distinctive semantic functions which are different from those of singular terms and quantifiers. A plural term is said to *plurally refer* to multiple objects in a different way than a singular term *singularly refers* to a single object: it stands in a one-many referential relation to multiple objects all at once, rather than the ordinary one-one referential relation to a single object, and is alleged to not commit its user to any one-over-many object that somehow comprises those multiple objects, such as the set or class of them. A plural quantifier is said to *plurally quantify* over the first-order domain of discourse, which consists of first-order (singular) objects, and plural primitivism contends that we can thereby quantify over multiplicities of first-order objects without requiring the existence of anything beyond the first-order objects. For example, $\exists xx \forall x(x \prec xx \leftrightarrow x \notin x)$ is interpreted to mean the following English plural construction:

(1) There are some sets such that any one of them is not a member of itself and such that any set that is not a member of itself is one of them.

Plural primitivism contends that since (1) does literally not claim the existence of any set of non-self-membered sets, the formula $\exists xx \forall x(x \prec xx \leftrightarrow x \notin x)$ does not ontologically commit us to the Russell set and, furthermore, is *trivially true*.[3]

One prominent axiom of plural logic is the axiom scheme of *plural comprehension* (PCA henceforth):

$$\exists x \varphi(x) \to \exists xx \forall x \big(x \prec xx \leftrightarrow \varphi(x)\big), \tag{PCA}$$

where φ is any formula of plural logic without xx free. According to the aforementioned interpretation of the plural glossary, this is interpreted to mean the following statement in English:

(2) If there is something that is φ, then there are some things such that any one of them is φ and such that anything that is φ is one of them,

which neither includes any mention of sets nor commits us to the existence of any set. There is known to be a 'canonical' mutual interpretation between the standard system PFO of plural logic and the system MSOL of monadic second-order logic. In view of this mutual interpretation, PCA corresponds to the second-order axiom (scheme) of impredicative comprehension of (monadic) second-order logic:

$$\exists X \forall x \big(Xx \leftrightarrow \varphi(x)\big), \tag{SCA}$$

where φ is any formula of MSOL without X free. This mutual interpretation suggests the so-called *plural interpretation* of (monadic impredicative) second-order logic, in which second-order variables are interpreted as plural variables (unless their values are not empty). Many advocates of plural primitivism regard PCA, or SCA under the plural interpretation, just as a trivial

[3] Resnik (1988) and Parsons (1990) object that despite appearance, we often need to understand plural terms as referring to sets or similar collection-like objects in order to process sentences containing plural terms. In this article, I proceed on the assumption of the ontological innocence of plural parlance and then aim to show that this assumption renders the plural primitivist case for PCA untenable—or, at least, unconvincing.

or *a priori* logical truth and takes this triviality of PCA as one of the main merits of plural primitivism.[4]

This article aims to critically examine PCA and offer a comprehensive argument that, while it can be true in many contexts and circumstances, the alleged logicality, a priority, and triviality of the truth of the full version of PCA are highly controversial.[5]

To conclude this introductory section, I introduce one notational convention. Plural primitivist parlance is sometimes difficult to express grammatically, unequivocally, and/or idiomatically in English. For example, 'some things are φ' or 'there are some things such that φ' is ambiguous: on the one hand, it may mean that there are two or more things each of which satisfies a singular predicate φ; on the other hand, it may also mean that there are some things that collectively satisfy a plural (collective) predicate φ. Moreover, as Resnik (1988) and Parsons (1990) note, the plural primitivist translation of universal plural quantification is clunky and hard to read. Accordingly, I will use the singular term 'plurality' to mean *many* in the plural primitivist sense—that is, what plural primitivists take plural nouns to denote. For example, to express the collective reading of 'some things are φ', I will use the singular construction 'there is a plurality that is φ'; I will render a universal plural quantification, $\forall xx\varphi(xx)$, as 'every plurality is φ' or 'for every plurality, φ', rather than the official (and awkward) plural primitivist paraphrase, 'it is not the case that there are some things such that it is not the case that φ'. However, the reader should always bear in mind that in such sentences, 'plurality' does not refer to any singular 'one over many' object that somehow 'comprises' several objects, such as a set, class, and platonist universal.

2. PCA as a schema of definitions

Elaborate justifications of PCA are scarce in the literature, perhaps reflecting the aforementioned common plural-primitivist view that its truth is trivial. Nonetheless, scattered remarks in the literature hint at why they think it is, and I will discuss them in turn. The first thought is that PCA is a schema of definitions. For example, Hossack (2014, 526) and Florio and Linnebo (2021, 229) write as follows:

> [U]nlike the other comprehension axioms, the plural axiom does not subserve ontology. Instead it subserves deduction, by underwriting our introduction of new denoting expressions. Given the Theory of Descriptions, [PCA] licenses us, whenever we are given a formula $\phi(u)$, to define 'the $\phi(u)$-ers' accordingly, provided at least one thing satisfies the formula $\phi(u)$. (Hossack, 2014, 526);
>
> Provided that a condition is well defined and has at least one instance, *of course* the condition can be used to define a plurality of all and only its instances. (Florio and Linnebo, 2021, 229)

[4] Florio and Linnebo (2021) write '[m]any philosophers regard [PCA] as utterly trivial and insubstantial'; we will see several examples of such philosophers shortly.

[5] The same or similar diagnoses are reached elsewhere in the literature via a variety of arguments; e.g., Resnik (1988), Parsons (1990), Hazen (1993), Linnebo (2003), and Hossack (2014), to list a few; Rumfitt (2018) also gave an interesting argument that PCA is inconsistent with a neo-Fregean abstraction principle for ordinals, by which he suggests that at most the Δ^1_1-fragment of PCA can be logically true. Some of my arguments resemble or overlap with theirs, and I will return to them in due course. Hossack (2014) and Florio and Linnebo (2020, 2021) also propose alternatives to PCA: Hossack incorporates stratification into the axiom of comprehension, inspired by Quine's NF; Florio and Linnebo, rather inspired by the Zermelo-Fraenkel set theory, replace comprehension with separation. Florio and Linnebo's theory, called *critical plural logic*, is particularly relevant to the discussion of this article, and I will return to it in due course.

We note that both Hossack (2014) and Florio and Linnebo (2021) deny PCA and intend to characterize their opponents in these quotes.

For notational convenience, let $\imath x K(x)$ formally denote the definite singular description 'the thing (in the domain of discourse) that is K' or 'the K' for short, and $\hat{x}\, K(x)$ formally denote the definite plural description 'the things (in the domain of discourse) that are K'.[6] According to this view, PCA is regarded as a schema of definitions of plural terms $\hat{x}\, K(x)$ for predicates/formulae K of plural logic. In this section, I examine this *definitional* view of PCA.

2.1. PCA *as a schema of naming*

But what does it mean for a term $\hat{x}\, K(x)$ to be *defined* via PCA? We ought to answer this question to assess whether PCA is justifiable as a schema of definitions. Boolos, an arch plural primitivist, expresses the following thought about PCA as a schema of definitions:

> Like the familiar condition: $\exists x \forall y(Ky \leftrightarrow y = x)$ which must be satisfied by a definite singular description 'The K' for its use to be legitimate, there is an analogous condition that must be satisfied by definite plural descriptions. In the simplest case, in which a definite plural description such as 'the present kings of France' is the plural form of a definite singular description, the condition amounts only to there being one object or more to which the corresponding count noun in the singular description applies. . . . Thus like the definite singular description "The K," which has a legitimate use iff the K exists, i.e. iff there is such a thing as the K, "The Ks" has a legitimate use iff the Ks exist, i.e. iff there are such things as the Ks, iff there is at least one K. (Boolos, 1985, 164–5)

It is widely accepted that the existence of a *unique* K justifies the legitimacy of the definite singular description 'the K'. Boolos draws an analogy and contends that the existence of *at least one* K justifies the legitimacy of the definite plural description 'the Ks' in precisely the parallel way. However, this analogy is inappropriate.

The condition $\exists x \forall y(Ky \leftrightarrow y = x)$ for the legitimacy of the singular term 'the K' not only guarantees the non-emptiness of the condition K. It also assures us, by virtue of the initial existential quantifier '$\exists x$', that the range of first-order quantifiers, that is, the domain of all possible referents of singular terms, contains a (unique) entity to be named 'the K'—or any other singular term—that has the desired property K. Hence, the noun phrase 'the K' can be viewed as only *naming* something that has already been presented to us by the holding of the condition $\exists x \forall y(Ky \leftrightarrow y = x)$, and we can legitimately regard the following definitional schema of $\imath x K(x)$ as a *naming* principle:

(3) $\exists! x K(z) \to \forall z\big(z = \imath x K(x) \ \leftrightarrow\ K(z)\big)$, for all predicates $K(x)$.

Boolos draws an analogy and claims that whenever there are one or more K, then 'the Ks' is a legitimate plural noun phrase *plurally referring to* all and only objects that are K. For him, the following is also a legitimate naming schema:

[6] The term 'definite plural description' is ambiguous in the context of plural logic, since it can be interpreted to mean either the unique plurality that satisfies the condition in question, or the plurality that consists of all and only the things that satisfies the condition. Oliver and Smiley (2016) call the former type of a definite plural description a 'plurally unique description' and the latter type an 'exhaustive description'. Throughout this article, I use 'definite plural description' to mean Oliver and Smiley's exhaustive description.

(4) $\exists x K(z) \to \forall zz\big(zz = \hat{x}\, K(x) \;\leftrightarrow\; \forall z(z \prec zz \leftrightarrow K(z))\big)$;

or, equivalently under the axiom of plural extensionality,

(5) $\exists x K(z) \to \forall z\big(z \prec \hat{x}\, K(x) \;\leftrightarrow\; K(z))\big)$,

from which PCA follows by existential generalization. This (5) is a formalization (in plural logic) of the following principle that Boolos alleged as a truism in the quote above:

(6) If there is some thing that is K, then 'the Ks' is a legitimate term so that anything that is K is among what 'the Ks' plurally refers to, and that anything that is among what 'the Ks' plurally refers to is K.

However, the non-emptiness condition $\exists x K x$ alone does not assure, unless PCA is presupposed, that a plural variable, such as zz in (4), can have a value to be named $\hat{x}\, K(x)$ that meets the desired property $\forall z\big(z \prec \hat{x}\, K(x) \leftrightarrow K(z))\big)$. In a manner of speaking, the legitimacy condition $\exists x K x$ in (4) and (5) for a definite plural description $\hat{x}\, K(x)$ achieves only a half of the job that the legitimacy condition $\exists x \forall y(K y \leftrightarrow x = y)$ in (3) for a definite singular description undertakes, and the other half is only achieved by PCA. This yields a substantial difference. The principle (3) is *conservative*: it adds to no logical truth that has no occurrence of the defined term $\imath x K(x)$, and $\imath x K(x)$ is always eliminable in a deduction of a logical truth including no $\imath x K(x)$. By contrast, (5) is *not* conservative: in particular, they imply the instance of PCA for K, in which $\hat{x}\, K(x)$ does not occur. In my opinion, Boolos's analogy is only a weak analogy and falls short of justifying PCA as a naming principle.

Indeed, the alleged parallelism between (3) and (4) is not acceptable from some foundational standpoints. For example, predicativists deny the definite totality of all subsets of ω and reject the legitimacy of quantification over them. However, they accept many subsets of ω, such as $\{0\}$ and ω itself. Hence, under the assumption of (6), the definite plural description 'the things that are subsets of ω'—or, more simply, 'the subsets of ω'—would count as legitimate definite plural descriptions even for predicativists. But it should not. For, otherwise, although this would not commit predicativists to the existence of any special *object*, such as the powerset of ω, it would still license them to quantify over all subsets of ω via locutions such as 'any one of the subsets of ω' and 'some one of the subsets of ω'. Hence, from a predicativist point of view, (4) and (6) are highly controversial.

2.2. PCA *as a schema of definitions of plural membership*

Another possible understanding of PCA as a schema of definitions is to view (5) as making a definition of the *plural membership relation* $t \prec \hat{x}\, K(x)$. At first glance, one might find it reasonable to say that $K(z)$ 'defines' what it is that $z \prec \hat{x}\, K(x)$ via the biconditional $\forall z(z \prec \hat{x}\, K(x) \leftrightarrow K(z))$.

However, the term 'define' here cannot be understood as an explicit definition in the usual sense that it introduces a mere abbreviation $t \prec \hat{x}\, K(x)$ of $K(t)$. As we have noted in §2.1, the introduction of the term $\hat{x}\, K(x)$ together with its 'definitional' clause $\forall z(z \prec \hat{x}\, K(x) \leftrightarrow K(z))$ yields new logical truths that do not contain $\hat{x}\, K(x)$; namely, the addition of $\hat{x}\, K(x)$ is not conservative. One might still find it reasonable to say that $K(z)$ 'defines' what it is that $z \prec \hat{x}\, K(x)$ in some informal, intuitive sense or might try to give some formal definition of a 'definition' in this sense. However, no matter how the term 'define' would be defined (either informally or formally), the very idea that $K(z)$ defines what it is that $z \prec \hat{x}\, K(x)$ is vulnerable to the familiar 'vicious circle' argument.

For example, consider Frege's definition of the concept of natural numbers:

$$\forall X\big(\big(0 \in X \wedge \forall z(Xz \to X(z+1))\big) \to Xx\big).$$

Let $\Psi_F(x)$ denote this formula. When we apply the plural interpretation to $\Psi_F(x)$, it is translated into the following predicate:

> It is not the case that there are some things such that 0 is one of them, that if anything is one of them then so is its successor, and that x is not one of them.

Let us call this predicate $\psi_F(x)$. According to the definitional reading of (5) in question, $\psi_F(z)$ defines what it is that $z \prec \hat{x}\, \psi_F(x)$. Then, whether a given object z is one of $\hat{x}\, \psi_F(x)$ should be determined by following the definition of $z \prec \hat{x}\, \psi_F(x)$. However, this is not possible. The membership relation $z \prec \hat{x}\, \psi_F(x)$ is defined as the satisfaction of ψ_F by z. Hence, to determine whether $z \prec \hat{x}\, \psi_F(x)$, we have to determine whether $\psi_F(z)$. To determine it, we take an arbitrary plurality yy and then determine whether yy is inductive and whether $z \prec yy$. However, it can be the case that yy happens to coincide with $\hat{x}\, \psi_F(x)$. Hence, this process involves determining whether $z \prec \hat{x}\, \psi_F(x)$, which is exactly what we originally wanted to do. By this familiar kind of an argument, (4) should not be read as defining the predicate '$z \prec \hat{x}\, K(x)$': in simple words, this is because the definiens refers to the definiendum in such a definition.[7]

3. PCA as a trivial truth

Some plural primitivists regard PCA as a truism. I have argued that PCA cannot be justified as a consequence of a schema of definitions. Those plural primitivists might instead think the other way around: it is not that PCA is justified by the legitimacy of the definite plural descriptions $\hat{x}\, K(x)$s, but rather that the legitimacy of $\hat{x}\, K(x)$s follows from the self-explanatory truth of PCA. This suggests the most direct justification of PCA, namely, that it is a trivial truth. For example, Lewis and Boolos are among those plural primitivists who regard PCA as such:

> Examples to show the evident triviality of a principle of plural 'comprehension': If there is at least one cat, then there are some things that are all and only the cats. (Regimented ... then there are some things such that, for all x, x is one of them iff x is a cat.) Likewise, if there is at least one set, then there are some things that are all and only the sets. (Lewis, 1991, p. 63)

> [T]he translation of the notorious $\exists X \forall x(Xx \leftrightarrow x$ is not a member of $x)$, where the first-order variables are taken to range over absolutely all sets is "(If there is a set that is not a member of itself, then) there are some sets that are such that each set that is not a member of itself is one of them and each set that is one of them is not a member of itself," as vacuous an assertion about sets as can be made, as desired. (Boolos, 1985, p. 76)

[7] Hossack (2014) makes a similar vicious circle argument against PCA.

Among others Hossack and Uzquiano also call PCA a 'harmless *a priori*' truth (Hossack, 2000, p. 422) and a 'evident triviality' (Uzquiano, 2003, p. 77).[8]

But why is PCA a trivial truth? As Frege (essentially) showed, PCA makes the existence of the least fixed-point of any positive operator a logical truth, which seems to be a highly non-trivial consequence. PCA and its canonical English translation (2) justify (5) and (6), respectively, as naming principles, but, as we have seen, they would license one to make definite plural reference to all and only subsets of ω, which is unacceptable for predicativists. There exist philosophers who seek an alternative to PCA, such as Hossack (2014) and Florio and Linnebo (2021, Ch. 12), and their proposals should not be dismissed by blaming them for missing a trivial truth. It rather seems to me a trivial truth that PCA is not a trivial truth. Having said that, for the sake of the subsequent argument, let us try to examine the alleged triviality of the truth of PCA in more depth.

Why do Boolos, Lewis, and others take PCA to be trivially true? The underlying thought seems to be as follows: any non-empty predicate K *determines* a plurality by prescribing its (plural) membership condition, under which all and only things that satisfy K have that membership. Once such a plurality is determined and given to us, we can name it $\hat{x}\,K(x)$ (or whatever name one likes).

However, a 'vicious circle' argument similar to the one made in §2.2. can be raised against the thought in question, according to which the plural membership of $\hat{x}\,K(x)$ is determined by the predicate $K(z)$. Suppose $K(z)$ is of the form $\forall yy\psi(x, yy)$ with a universal plural quantifier (e.g., $\psi_F(z)$ taken in §2.2.). Take any object a. Since $\psi(a, \hat{x}\,K(x))$ is a substitution instance of a direct sub-formula of K, it appears to be sensible to say that whether $K(a)$ or not is partly determined by whether $\psi(a, \hat{x}\,K(x))$ or not. However, $\psi(z, yy)$ may contain a sub-formula of the form $t \prec \hat{x}\,K(x)$ for various terms t. Hence, whether $\psi(a, \hat{x}\,K(x))$ or not is also partly determined by whether various objects have the membership of $\hat{x}\,K(x)$ or not. We thereby get involved in a vicious circle of determination, in which the determinans refers to the determinatum.[9]

This typical pattern of the vicious circle argument applies to other attempts to legitimize a plurality that appeal to other justificatory relations in which the plurality stands to some predicate K. For example, one might claim that $\hat{x}\,K(x)$ has a definite plural membership *because* K is definite so that it definitely demarcates the domain of discourse into two realms, that is, the things that are K and those that are not. Let $K(x)$ be $\forall yy\psi(x, yy)$ as before. We may, then, ask why K is definite? One sensible answer is that it is *because* $\psi(z, yy)$ is definite no matter what plural referents are assigned to the plural variable yy. Hence, K is definite partly because $\psi(x, \hat{x}\,K(x))$ is definite. We may keep asking why, and, as before, the answer may ultimately be that it is partly *because* $\hat{x}\,K(x)$ has a definite plural membership. One might alternatively say that the definiteness of the plural membership of $\hat{x}\,K(x)$ is *explained by*, or *reduced to*, the definiteness of K. Then, similarly, the definiteness of K is partly explained by, or reduced to, the definiteness of $\psi(z, \hat{x}\,K(x))$ and, ultimately, the definiteness of the plural membership of $\hat{x}\,K(x)$. We get involved in a vicious circle of reasons, explanations, or reductions.

I admit that these are sloppy arguments. For the first vicious circle argument, it can be objected that the sense in which K is said to determine the membership of $\hat{x}\,K(x)$ is different from the sense in which the substitution instance $\psi(a, \hat{x}\,K(x))$ is said to (partly) determine $\forall yy\psi(a, yy)$ (i.e., $K(a)$). Similarly, it can be objected that the sense in which the definiteness

[8] Hossack later changed his view in (Hossack, 2014).

[9] Linnebo (2018) raises essentially the same argument against SCA in terms of grounding.

of $\hat{x}K(x)$ is explained by, or reduced to, the definiteness of $K(x)$ is different from the sense in which the definiteness of $K(x)$ is (partly) explained by, or reduced to, the definiteness of $\psi(x, \hat{x}K(x))$; one can raise a similar objection regarding the sense of 'because'. All these are fair objections: the vicious circle arguments at stake employ the terms like 'determine', 'explain', 'reduce', and 'because' without sufficiently delineating their meanings.

Nevertheless, these vicious circle arguments raise one general concern about justifications of PCA of the kind at stake. Recall that these justifications, as well as that of PCA discussed in §2.2., appeal to a certain justificatory relation in which the plural membership of a plurality xx stands to some predicate K prescribing the (plural) membership condition of xx, under which all and only things that satisfy K have the membership. This amounts to the idea that pluralities are justified as the *extensions* of predicates —or the extensions of what those predicates denote, such as propositional functions, if one does not want to let linguistic expressions in themselves to have extensions. However, didn't we learn the lesson from the foundational debate over mathematics in the early 20th century that this idea does not provide an adequate justification for impredicative comprehension?

Parsons (2002) concisely characterizes Russell's and Weyl's predicativism as the view that 'what are called sets are extensions of concepts' (p. 374); essentially the same view is shared by Poincaré and Frege. As Goldfarb (1989) points out, the need for ramification in Russell's theory does not come from any constructive idea about sets (or 'classes' in Russell's own terms), but from his thought that logic concerns propositional functions rather than sets. A set may be given different specifications belonging to different levels in Russell's ramified hierarchy, but its identity is solely determined by its members; hence, if Russell's theory were a theory of sets, then it would require no ramification. By contrast, identity of a propositional function is not determined by the objects of which it is true, but by way of the manner in which it is 'presented' (in terms of Goldfarb, 1989). Different presentations correspond to different propositional functions, even if they are true of exactly the same objects; in terms of Quine's famous example, 'is an animal with kidneys' and 'is an animal with a heart' present two distinct propositional functions while yielding the same set. For Russell, a set (if it exists) is the extension of a propositional function, which gives a specification of the members of the set, and the membership relation is defined in terms of the satisfaction of the propositional function.[10] Similarly, Weyl denies the realist view that Bernays (1983) later dubbed the 'quasi-combinatorial' concept of sets,[11] and advocates that *infinite* sets can only be justified as extensions of properties that can be constructed ('derived' from 'primitive properties') in a certain manner.[12] Hence, while they do not reject sets outright, both Russell and Weyl view a set as determined by, and auxiliary to, something else that prescribes a condition for a thing to be a member of the set. For Russell, the set membership is defined/determined/explained in terms of the satisfaction of propositional functions; for Weyl, it is defined/determined/explained in terms of the exemplification of those constructible properties. When they are asked why a certain set exists, their answer would be that it is because such and such a propositional function or property exists.

10 Hence, Russell's theory does not primarily concern sets. He did not take sets as the most fundamental entities of logic, but as something for more practical purposes; for example, Russell says 'the chief purpose which classes serve, and the chief reason which makes them linguistically convenient, is that they provide a method of reducing the order of a propositional function [without affecting the truth or falsehood of its values]' (Russell, 1908, 242).

11 See (Weyl, 1918, p. 23), for example.

12 See (Weyl, 1918, §4–§8), for example.

A justification of PCA of the kind at stake goes parallel. It says that a plurality is determined by some predicate K that prescribes its (plural) membership condition; it says that a given object a is (or is not) a plural member of the Ks because a satisfies (or does not satisfy) K. However, a plurality xx could be justified as the extension of a predicate K, only when K is a legitimate/meaningful predicate. Russell's, Weyl's, and other predicativists' criticism of impredicative comprehension concerns the very legitimacy/meaningfulness of impredicative predicates K—or the existence of the propositional function or property that K 'presents' (in terms of Goldfarb, 1989 in his exposition of Russell) or 'expresses' (in terms of Weyl). Hence, merely associating a plurality with a predicate K does not address the predicativists' concern about impredicative comprehension. A natural question, then, is why a justification of the kind at stake succeeds for PCA while the same kind of justification is widely considered inadequate for SCA. Vicious-circle arguments of the familiar sort we have just discussed have driven many philosophers and logicians toward predicativism. Why, then, can only PCA resist them? Even if it can, this is not a trivial matter and would call for a non-trivial argument.

4. The Gödel-Bernays realism

From the discussion in §2.2. and §3, I tentatively conclude that we should not take each instance of PCA,

$$\exists x\varphi(x) \to \exists xx\forall x\big(x \prec xx \leftrightarrow K(x)\big),$$

to be true by virtue of any reductive, explanatory, or determination relation in which the plural membership $x \prec xx$ stands to the predicate K, and that we cannot justify PCA by regarding pluralities as extensions of predicates. This (tentative) conclusion naturally suggests that a plausible justification of PCA requires that what pluralities exist, and what their plural members are, should be determined independently of us and any linguistic description by us. This view may be called 'realism' about pluralities. Indeed, realism is a standard strategy to justify impredicative definitions in set theory. Gödel famously appealed to realism of sets in his defense of impredicative definitions in set theory. While he acknowledges that the construction or definition of a thing 'can certainly not be based on a totality of things to which the thing to be constructed belongs' (Gödel, 1983, p. 136), he suggests that

> If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members which can be described . . . only by reference to this totality. (*ibid*.)

The idea being considered here is 'realist' only in the sense that the extension of the plural membership relation $\prec$, the range of plural (or second-order) quantifiers, and the truth values of plural sentences are objectively determined independently of us; it should not be confused with the idea that there exists an object, independently of us, that can be a single referent of a plural term, and the range of plural quantifiers consists of some such objects, which would collapse plural primitivism into singularism.

However, the crucial problem with this 'realist' conception of plurals is that it completely dissociates plural membership relation $\prec$ from various possible conditions K, whereby the alleged *a priori* connection between the membership of $\hat{x}K(x)$ and its description $K(z)$ is totally lost. With the 'realism' of pluralities, a condition or specification, like K, is a mere means of picking out a plurality from the domain of plural quantifiers *if such a plurality belongs to that* *domain*. We need an extra argument to guarantee, for each condition K, that all and only objects that are K stands in the independently given relation $\prec$ to *some* plurality. To my understanding, the same line of consideration led Bernays to propose the 'quasi-combinatorial' conception of sets in justification of impredicative definitions in set theory, according to which a set is viewed as

> the result of infinitely many independent acts deciding for each number whether it should be included or excluded. (Bernays, 1983, p. 260)

Namely, the set $\{z \in x \mid K(z)\}$ exists as 'the result of infinitely many independent acts of' including objects in the set only when K is true of them (and excluding objects from it only when K is false of them).[13]

Can we adapt the Gödel-Bernays 'realist' argument for a justification of PCA as a logical axiom? On the one hand, it does not seem to be straightforward to adapt the Gödelian realism for a justification of PCA. Pluralities are actually not such objects that exist, or do not exist, in the ordinary sense according to the plural primitivist thesis of the ontological innocence of plural terms. Hence, the desired adaptation requires to spell out what it means that pluralities and the totality of them objectively exist independently of us.

On the other hand, Bernays's notion of 'quasi-combinatorial' definitions of sets can be relatively straightforwardly adapted to definitions of pluralities. A plurality can then be understood as the outcome of possibly infinitely many independent acts of picking objects and deciding whether to include them in or exclude them from that plurality.

In my opinion, however, the Bernaysian 'quasi-combinatorial' conception of pluralities hardly justifies PCA *as a logical axiom*. It requires infinitary acts of selecting things. Set theory is a descriptive science of a specific subject matter, that is, sets and their universe. Metaphorically speaking, even though we humans cannot perform infinitely many acts, God can do it in His creation of the universe of sets, and that universe may well contain the results of such infinitary acts by God. Logic, by contrast, is not a descriptive science of any specific subject matter; rather, it is supposed to be topic-neutral and universally applicable, studying valid inferences that preserve truth in all possible circumstances. I find no compelling reason to assume that the domains of discourse in all those circumstances involve the results of such infinitary acts: God may decide not to exercise His ability to perform infinitary acts in creating some of those circumstances. Furthermore, to me, logic is the study of reasoning by us, not by God, and thus logic should not postulate an axiom that presupposes something that we can never perform even in ideal circumstances.

After all, PCA under the Gödel-Bernays realist conception of pluralities is apparently no more logical than the axioms of separation in set theory and can hardly be called a logical axiom.[14]

[13] Linnebo (2003, §IV) argues that PCA requires this Gödel-Bernays realism, and maintains this view also in (Florio and Linnebo, 2021), where they write 'To say that plural comprehension is permissible on a condition φ is to say that we may reason quasi-combinatorially about all the φs' (p. 229).

[14] Florio and Linnebo (2021, Ch. 10) present an argument for the quasi-combinatorial conception of pluralities, which relies on the assumption of the *traversability* of pluralities. However, formulating this assumption requires infinite disjunctions of arbitrarily large cardinality and thus, in my view, covertly commits us to infinitary acts (of articulating infinitely long conditions). Their argument provides a basis for *critical plural logic*, their theory of *circumscribed* pluralities, in which pluralities are conceived of as extensionally definite, modally rigid, and traversable. However, they themselves do not regard it as a logic in the sense at issue in this article; see (Florio and Linnebo, 2021, Ch. 12.7).

5. Semantics

There might be other, more plausible 'realist' justifications of PCA as a logical truth than the Gödel-Bernays realism, although I am not currently aware of any. In this section, rather than exploring such realist justifications further, I turn to a crucial presupposition of the Bernaysian 'quasi-combinatorial' conception, namely, the *definiteness* of impredicative plural formulae. An infinitary selection of the members of a plurality could not be carried out without the definiteness of a condition K according to which the members are selected. More generally, the definiteness of K seems to be a necessary condition for any justification of the definiteness of the definite plural description 'the Ks'. Furthermore, if we succeed in justifying the definiteness of K without being committed to a vicious circle, then we can simply legitimize 'the Ks' as a term plurally referring to the extension of K. However, the claim that every formula/predicate is definite has never been unanimously accepted in the history of the philosophy of mathematics; for example, intuitionism, strict finitism, and constructivism reject it. This is why Gödel appealed to realism about sets and their totality to guarantee the definiteness of conditions K, in justifying impredicative definitions in set theory. In what follows, I present a problem with the alleged definiteness of impredicative plural formulae from a semantic point of view.

5.1. Semantic Determinacy and Commitment

Let $\mathfrak{M}$ be a first-order structure for a first-order language $\mathcal{L}$. The extensional meanings of $\mathcal{L}$-expressions, such as the truth values of $\mathcal{L}$-sentences and the referents of closed $\mathcal{L}$-terms, are determined solely by $\mathfrak{M}$. Now, suppose we augment $\mathcal{L}$ with a new first-order predicate P and constant c, and let us call the thus extended first-order language $\mathcal{L}'$. The referents of closed $\mathcal{L}'$-terms including c is not determined solely by $\mathfrak{M}$, and nor is the truth value of $\mathcal{L}'$-sentences including P. This is because $\mathfrak{M}$ tells us nothing about P and c. We may describe this situation as the *semantic determinacy* of $\mathcal{L}$ in $\mathfrak{M}$ and the *semantic indeterminacy* of $\mathcal{L}'$ in $\mathfrak{M}$. To make $\mathcal{L}'$ *semantically determinate*, we typically augment $\mathfrak{M}$ with extra semantic information, namely, fixed interpretations of P and c.

For a more relevant example, let $\mathcal{L}^2$ be the second-order extension of $\mathcal{L}$ which augments $\mathcal{L}$ with second-order variables and quantifiers. Under Henkin semantics, $\mathcal{L}^2$ is not semantically determinate in $\mathfrak{M}$. To make it semantically determinate, we need to augment $\mathfrak{M}$ with a set of subsets of the first-order domain of $\mathfrak{M}$ as the range of second-order quantifiers. By contrast, $\mathcal{L}^2$ *is* semantically determinate in $\mathfrak{M}$ alone under the standard ('full') semantics of second-order logic, in which second-order quantifiers are automatically interpreted to range over absolutely all subsets of the first-order domain of $\mathfrak{M}$: no new information beyond the specification of that first-order domain is required.

Now, under Henkin semantics, we may call a second-order language *semantically further-committal* (relative to first-order logic), in the sense that its semantic determinacy requires further *semantic* information beyond that supplied by the semantic interpretation of its first-order part. By contrast, under the standard semantics, a second-order language is not semantically further-committal: a first-order model-theoretic structure solely determines the semantics of all its expressions and, in particular, make all its predicates definite.

Semantic commitment, in this sense, is closely related to ontological commitment, but the two notions are independent. Under Henkin semantics, in order for $\mathcal{L}^2$-expressions to receive definite extensional meaning, further objects beyond those supplied by $\mathfrak{M}$—namely, a set of subsets of the domain of $\mathfrak{M}$ and its members—are required, and thus we may say that $\mathcal{L}^2$ is ontologically further committed to these objects. Hence, a second-order language under Henkin semantics is both semantically and ontologically further-committal. By contrast, a second-order language under the standard semantics is only ontologically further-committal and not semantically further-committal. The converse does not necessarily hold either: further semantic commitment does not entail further ontological commitment. For example, Florio and Linnebo (2016) propose the *plurality-based Henkin semantics* of plural logic, in which a plural language is interpreted by a first-order structure augmented with a *super-plurality* over the first-order domain—that is, a plurality of pluralities of first-order objects—as a range of plural quantifiers: a plural quantifier is then interpreted as ranging over the plural members of that super-plurality.[15] A superplurality is a plurality of pluralities of individual objects and alleged to incur no ontological commitment beyond those individual objects. Hence, under the plurality-based Henkin semantics, plural expressions are considered to be not ontologically further committal, while they are semantically further committed to that super-plurality.[16]

5.2. The Maximum Domain Thesis

Is a language of plural logic semantically further-committal or not? Most plural primitivists seem to think that it isn't, and even take this semantic not-further-committalness as one of the main virtues of plural primitivism. For those plural primitivists, the range of plural quantifiers is automatically and uniquely fixed once the semantic interpretation of the first-order vocabulary is fixed. Let us call this view the *unique domain thesis*. This can be compared to the semantic treatment of the logical relation of identity: its extension is uniquely fixed once the semantic interpretation of the (non-logical) first-order vocabulary is fixed.

But what should such a uniquely determined range be like? Recall that the standard semantics of second-order logic renders a second-order language semantically not-further-committal because it automatically interprets second-order quantifiers to range over *absolutely all* subsets of the first-order domain. Many plural primitivists draw an analogy from the standard semantics of second-order logic, and take plural quantifiers as ranging over *absolutely all* pluralities of first-order objects. Let us call this idea the *maximum domain thesis*. The unique domain thesis does not imply the maximum domain thesis, but the unique domain thesis without the maximum domain thesis appears to be unnatural.[17] Indeed, the maximum domain thesis is quite common among plural primitivists.[18]

5.3. Interdependence with sets

In this sub-section, however, I argue that the maximum domain thesis is in tension with the alleged logicality of the plural vocabulary and thus of PCA.

15 For discussions of superpluralities, see also (Hazen, 1997), (Oliver and Smiley, 2005, 2016), (Rayo, 2006), (Linnebo and Nicolas, 2008), and (Florio and Linnebo, 2021).

16 Florio and Linnebo (2016) offer another option for the range of plural quantifiers in a plurality-based Henkin model, that is a plural property: then, plural quantifiers are interpreted as ranging over the pluralities of first-order objects that satisfy that plural property.

17 I am aware of only one natural alternative, namely, the *predicativist* domain thesis that plural quantifiers range over all and only pluralities describable by formulae with no plural quantifiers; however, this alternative clearly fails to justify PCA anyway.

18 According to Florio and Linnebo (2016), 'nearly all writers who have embraced plural logic on the plurality-based semantics ascribe to this system metalogical properties which presuppose that the semantics is standard rather than Henkin' (p. 566). Note that Florio and Linnebo (2016) themselves do not even embrace the unique domain thesis.

5.3.1. Etchemendy's argument

We begin with Etchemendy's (1990) well known argument. He contends that the Tarskian definition of logical consequence is inappropriate for second-order logic under the standard semantics, because it renders many distinctively mathematical statements, such as the continuum hypothesis (CH), either logically true or logically false. It is well known that, under the standard semantics, there are formulae $N(X)$ and $R(X)$ of some language of MSOL that pin down (categorically axiomatize) the domains $\mathbb{N}$ and $\mathbb{R}$ of the standard models of arithmetic and real ordered field (up to isomorphism), respectively. Using these, one can construct a sentence Γ such that Γ is logically true in MSOL iff CH is true and that Γ is logically false in MSOL iff CH is false.[19] Hence, if MSOL under the standard semantics is a genuine logic, then CH is either logically true or logically false.

One might think that a similar objection applies to plural logic under the maximum-domain thesis, but the situation is not so simple. To illustrate this, for each formula Φ in MSOL, let $\Phi^{(p)}$ denote the canonical translation of Φ in PFO; conversely, for each formula φ in PFO, let $\varphi^{(1)}$ denote the canonical translation of φ in MSOL. Then, $\Gamma^{(p)}$ is a natural candidate for a PFO-sentence to express CH. However, the original Γ is equivalent to CH in MSOL under the (set-based) standard semantics because second-order quantifiers are interpreted as ranging over sets. Given that the plural quantifiers in $\Gamma^{(p)}$ do not range over sets, $\Gamma^{(p)}$ need not be equivalent to Γ: without further assumptions, it can be the case that there is a plurality yy such that no set Y is coextensive with yy; it can equally be the case that there is a set Y such that no plurality yy is coextensive with Y.

Of course, Γ and $\Gamma^{(p)}$ are equivalent under the assumption that sets necessarily coincide with pluralities—namely, that, necessarily, for every set X there is a plurality of the members of X, and that, necessarily, for every plurality xx, there is a set of the plural members of xx—, but this assumption appears to be unmotivated from the plural primitivist point of view. First, if pluralities necessitate coextensive sets, then they are naturally taken to carry ontological commitment to sets. Second, plural primitivists aim to offer a set-free alternative logic that is still as strong as second-order logic, so they would want to avoid positing such a tight logical/metaphysical connection between pluralities and sets. Third, the assumption entails that only set-sized pluralities exist, thereby precluding the plural interpretation of proper classes.

Furthermore, even when xx and X are coextensive, $N^{(p)}(xx)$ need not be equivalent to $N(X)$. Again, without further assumptions, it can be the case that there is a plurality yy that is not coextensive with any set Y but witnesses the ill-foundedness of xx; it can equally be the case that there is a set Y that is not coextensive with any plurality yy but witnesses the ill-foundedness of X. Hence, either or both of $N^{(p)}(xx)$ and $N(X)$ may fail to pin down $\mathbb{N}$. Similarly, either or both of $R^{(p)}(xx)$ and $R(X)$ may fail to pin down $\mathbb{R}$.

Even though $\Gamma^{(p)}$ may not be equivalent to CH, and even though $\Gamma^{(p)}$ may not be about the genuine $\aleph_0$ and $\aleph$, it still feels like a mathematical statement. However, given that $\Gamma^{(p)}$ lacks any logical connection with CH, it is less clear whether it should be treated as an extra-logical statement. Indeed, many apparently distinctively mathematical statements—for example, that any group of order 361 ($= 19 \times 19$) is abelian—are logical truths even in first-order logic. Since

[19] See (Etchemendy, 1990, Ch. 9, note 11). In MSOL, we actually need an extra non-logical assumption—for example, an assumption that enables us to code an ordered pair of any two objects—to construct such a sentence Γ. Jané (2005) presents essentially the same line of argument against second-order logic under the standard semantics.

plural primitivists aim to drastically strengthen logic via plural vocabulary, they might be prepared to treat more statements of more mathematical flavour, including $\Gamma^{(p)}$, as logically true (or logically false).

In what follows, I supplement Etchemendy's argument by presenting two further cases in which, under reasonable assumptions, certain sentences of plural logic stand in a more direct interdependence with distinctively mathematical statements.

5.3.2. Case 1

Let $\mathcal{L}^2_{\mathbb{N}}$ and $\mathcal{L}^p_{\mathbb{N}}$ denote the languages of second-order and plural arithmetic, respectively, whose non-logical vocabulary is exactly that of the first-order language $\mathcal{L}_{\mathbb{N}}$ of arithmetic. We denote the standard model of arithmetic by $\mathfrak{N}$, which is a structure for both $\mathcal{L}^2_{\mathbb{N}}$ (under the standard semantics) and $\mathcal{L}^p_{\mathbb{N}}$ (under the maximum domain thesis).

We define the classes Π^p_n and Σ^p_n $(n \in \mathbb{N})$ of $\mathcal{L}^p_{\mathbb{N}}$-formulae by the obvious analogy with the classes Π^1_n and Σ^1_n $(n \in \mathbb{N})$ of $\mathcal{L}^2_{\mathbb{N}}$-formulae: every $\mathcal{L}^p_{\mathbb{N}}$-formula without plural quantifiers is $\Pi^p_0 = \Sigma^p_0$; if $\varphi(xx)$ is Σ^p_n, then $\forall xx\varphi(xx)$ is Π^p_{n+1}; if $\varphi(xx)$ is Π^p_n, then $\exists xx\varphi(xx)$ is Σ^p_{n+1}; then, $\Phi^{(p)}$ is equivalent (in PFO) to a Σ^p_n-formula for any Σ^1_n-formula Φ, and $\varphi^{(1)}$ is equivalent (in MSOL) to a Σ^1_n-formula for any Σ^p_n-formula φ.

According to the maximum domain thesis, $\mathfrak{N} \models \forall xx\varphi(xx)$ holds if and only if $\mathfrak{N} \models \varphi(xx)$ for absolutely all pluralities xx of natural numbers. But what pluralities are included in that range? Take any *set* X of natural numbers, which belongs outside the domain $\mathbb{N}$ of $\mathfrak{N}$. Consider the definite plural description 'the things that are members of the set X'. This description is *predicative*—in the sense that it contains no plural quantifiers—in a language that can express the membership relation and whose domain of discourse contains X. Hence, it is much more harmless and reasonable to treat it as a legitimate definite plural description than impredicative ones. Hence, although X belongs outside $\mathbb{N}$, it exists anyway, and it seems plausible to me, under the maximum domain thesis, that the plurality of the members of X also exists. More generally, the following assumption appears to be quite plausible:

(7) For every set X, there is a plurality of the members of X.

Let $\mathfrak{A}$ be any first-order structure with domain A. Under the assumption of (7), the range of plural quantifiers in $\mathfrak{A}$ subsumes the powerset of A.

We next assume the following:

(8) The least infinite ordinal ω in the universe of sets is isomorphic to the domain $\mathbb{N}$ of the standard model of arithmetic.[20]

Under the assumptions (7) and (8), if a Σ^1_1-sentence Φ is true in the universe of sets, then $\Phi^{(p)}$ is also true in $\mathfrak{N}$. For example, let Ψ be a Σ^1_1-sentence expressing that there is a set of natural number that codes a (countable) model of ZFC plus one inaccessible cardinal. If the universe of sets is a model of ZFC and contains two inaccessible cardinals, then Ψ is true in the universe of sets (by the reflection principle and Löwenheim-Skolem theorem), from which it follows that $\mathfrak{N} \models \Psi^{(p)}$. By contraposition, if $\mathfrak{N} \not\models \Psi^{(p)}$, then there is at most one inaccessible cardinal in the universe M of sets (if the universe of sets is a model of ZFC).

20 If we work within a set-theoretic meta-theory, (8) means that the model of object set theory is an ω-model of set theory.

Furthermore, many plural primitivists (e.g., Hossack, 2000; McKay, 2006, Ch. 6; Oliver and Smiley, 2016, Ch. 13.2) hold the following:

(9) Plural logic can pin down (categorically axiomatize) the standard model $\mathfrak{N}$ of arithmetic;

they often take (9) to be one of the main advantages of plural logic over first-order logic. Let $\varXi$ be a sentence of plural logic that pins down $\mathfrak{N}$. Under the assumption of (9), the existence of a countable model of ZFC plus one inaccessible cardinal implies that $\varXi \wedge \varPsi^{(p)}$ is logically true. By contraposition, if $\varXi \wedge \varPsi^{(p)}$ is logically false, then the universe contains at most one inaccessible cardinal (again if the universe of sets is a model of ZFC).

This may be taken to indicate that which pluralities exist, or which sentences are logically true in plural logic, depends on which sets exist, or conversely that which sets exist depends on which pluralities exist or which sentences are logically true in plural logic. That said, $\varPsi$ and $\varPsi^{(p)}$ (or $\varXi \wedge \varPsi^{(p)}$) are still not equivalent, and it may be that $\varPsi^{(p)}$ is true and $\varPsi$ is false. In the next subsubsection, I present a stronger case.

5.3.3. Case 2

We are presently comparing plural quantification over a first-order domain with singular quantification over subsets of the same domain, allowing for the possibility that the two are substantially different. To proceed with the current argument in a formally precise way, we need to fix a meta-theory in which the semantics for both types of quantification are defined. In what follows, we work within a sufficiently rich set-theoretic meta-theory and assume that plural logic (under the maximum domain thesis) receives the set-based Henkin semantics there. We denote the (genuine) least infinite ordinal ω in the meta-theory by $\mathbb{N}$, regarded as the domain of the standard model $\mathfrak{N}$ of arithmetic, and let $Pl(\mathbb{N})$ be some set of subsets of $\mathbb{N}$ serving as the range of plural quantifiers in $\mathfrak{N}$.

First, it seems reasonable to make the following assumption:

(10) The $\mathcal{L}^p_{\mathbb{N}}$-formula asserting the well-foundedness of a sub-plurality of $\mathbb{N}$—which is a $\varPi^p_1$-formula—is always correct in $\mathfrak{N}$;

that is, $(\mathbb{N}, Pl(\mathbb{N}) \cup \{\emptyset\})$ is a β-model of second-order arithmetic.

Next, we fix a model $\mathfrak{M} = (M, E)$ of set theory that is defined within the meta-theory. The domain M of $\mathfrak{M}$ will be regarded as 'the universe of sets' at the object level. I make two assumptions about $\mathfrak{M}$:

(11) $\mathfrak{M}$ is a well-founded model of some moderately strong theory T extending, say, the Kripke-Platek set theory KPω plus $\varSigma_1$-separation.

(12) The height of $\mathfrak{M}$ is sufficiently large so that the order-type of the ordinals in $\mathfrak{M}$ is greater than or equal to ω_1^L in the meta-theory;[21]

By (11), $\mathfrak{M}$ is isomorphic to some transitive model of T in the meta-theory (assuming that the meta-theory can prove Mostowski's collapsing theorem). Hence, in particular, $\omega^{\mathfrak{M}}$ is isomorphic to $\mathbb{N}$, and thus we will identify $\omega^{\mathfrak{M}}$ and $\mathbb{N}$ in what follows. Moreover, T is rich enough so that the constructible universe $L^{\mathfrak{M}}$ can be defined in its model $\mathfrak{M}$ and that many basic facts about

21 We can replace ω_1^L with a smaller ordinal, such as, the least stable ordinal.

it are true in $\mathfrak{M}$, such as the Shoenfield absoluteness. Both assumptions, (11) and (12), do not concern pluralities; rather, they are purely about the universe of sets (at the object level).

I will show that the truth values of any Σ^1_2-sentence Φ (in $\mathfrak{M}$) and its $\mathcal{L}^p_{\mathbb{N}}$-translation $\Phi^{(p)}$ (in $\mathfrak{N}$) coincide: that is, $(\mathbb{N}, Pl(\mathbb{N})) \models \Phi^{(p)}$ iff $(\mathbb{N}, \mathcal{P}(\mathbb{N})^{\mathfrak{M}}) \models \Phi$.

First, because PCA is a logical axiom and thus true in $\mathfrak{N}$, the $\mathcal{L}^p_{\mathbb{N}}$-translation $\mathsf{Z}_2^{(p)}$ of full second-order arithmetic Z_2 is true in $\mathfrak{N}$. Hence, in $\mathfrak{N}$, the constructible hierarhy (up to a certain level) can be coded in $\mathcal{L}^p_{\mathbb{N}}$. By (10), the codes of constructible sets in $\mathbb{N}$ are correct, and there is an ordinal ρ at the meta-level such that the union of the constructible sets that are coded in $\mathcal{L}^p_{\mathbb{N}}$ in $\mathfrak{N}$ coincides with L_ρ in the meta-theory. Furthermore, the Shoenfield absoluteness holds in $\mathfrak{N}$ in terms of these codes of constructible sets: any Σ^1_2-sentence Φ is true in in L_ρ, iff $\Phi^{(p)}$ is true in $\mathfrak{N}$.[22]

Second, by (11), the constructible hierarchy $L^{\mathfrak{M}}$ in $\mathfrak{M}$ coincides with the genuine one at the meta-level up to a certain level, say, L_τ.

Third, by (7), (10), and (12), the supremum of the order-types of well-orderings of $\mathbb{N}$ in $\mathfrak{N}$ is $\geq \omega_1^L$, and thus $\rho \geq \omega_1^L$. Moreover, it is also assumed that $\tau \geq \omega_1^L$ by (12).

Finally, take any Σ^1_2-sentence Φ. By the Shoenfield absoluteness in $\mathfrak{N}$, $\Phi^{(p)}$ is true in $\mathfrak{N}$, iff Φ is true in L_ρ. Since $\rho, \tau \geq \omega_1^L$, Φ is true in L_ρ, iff Φ is true in L_τ. By the Shoenfield absoluteness in $\mathfrak{M}$, Φ is true in L_τ, iff Φ is true in $\mathfrak{M}$.

There are many distinctively mathematical Σ^1_2-statements. For example, the existence of a countable transitive model of any recursive set theory, such as ZFC plus two inaccessible cardinals, is a Σ^1_2-statement. The axiom of Σ^0_n-determinacy ($n \in \mathbb{N}$) is also Σ^1_2.[23] These Σ^1_2-sentences may be undecidable in T: the existence of a countable transitive model of ZFC plus two inaccessible cardinals is independent of ZFC; Σ^0_n-determinacy for large n is independent of some weak theory T that meets the condition (11).[24] However, the truth of these statements in the universe M of sets are equivalent to the truth of its canonical $\mathcal{L}^p_{\mathbb{N}}$-translation in $\mathfrak{N}$. If we further assume (9), then they are either logically true or logically false. This mirrors Etchemendy's set up, in which a statement independent of the standard set theory ZFC (i.e., CH) comes out either logically true or logically false.

It might be objected that the assumptions (11) and (12) are ad hoc and that we need not accept them. However, they are assumptions purely about the universe of sets, asserting some desirable, or at least reasonable, property of the universe, and they do not concern pluralities. We can also cook up various different assumptions that yield similar consequences. The import of what we have discussed so far is that under the assumption of (7), certain additional assumptions purely about sets—e.g. (11) and (12)—and certain additional assumptions purely about pluralities—e.g. (10)—may together yield a strong interdependence between sets and pluralities.

5.3.4. One possible way out — the plurality-based Henkin semantics

These examples suggest that, under the maximum domain thesis, some logical truths in plural logic are interdependent with the ontology of sets and the structure of their universe. This

22 See (Simpson, 2009, Ch. VII.4) for the details of the coding of constructible hierarchy in second-order arithmetic as well as the proof of Shoenfield absoluteness in terms of that coding.
23 See (Simpson, 2009, V.8) for the definition of Σ^0_n-determinacy in the language of second-order arithmetic.
24 If the meta-theory decides these Σ^1_2-sentences, then $\mathfrak{M}$ also satisfy them by the condition (12).

consequence poses a challenge to the plural primitivists' claim that plural logic is a genuine logic.

One possible way to avoid this consequence is to adopt a semantics that is semantically *further-committal* but still ontologically not-further-committal, such as the aforementioned plurality-based Henkin semantics by Florio and Linnebo (2016). This might be an effective plural primitivist rejoinder to the argument I have just presented, but is not without problems. First, the notion of super-plurality (or plural property) is controversial. Second, it requires justification of the existence of a super-plurality (or plural property) that is closed under impredicative definitions of pluralities of first-order objects. Third, to make PCA a logical axiom under their semantics, all the models taken into account by the semantics need to satisfy PCA, but this requirement must be justified. Indeed, Florio and Linnebo give up PCA after all and propose what they call *critical plural logic* (Florio and Linnebo, 2020, 2021) in which PCA is severely restricted.[25]

6. Indescribable pluralities

The other possible plural primitivist rejoinder to the argument in §5.3. is to deny (7) and hold that the existence, or non-existence, of a plurality coextensive with a set X is completely independent of the existence, or non-existence, of X or, more generally, independent of any object of any kind that is not included in the first-order domain. However, then, my concern is that, without any connection with other collection-like objects, our ordinary understanding of plural reference and quantification seems to be hopelessly *uninformative* about to which pluralities we can plurally refer.

We apparently know the truth condition of, and can determine the truth value of, 'some boys are chatting at the corner', independently of any set or any object other than those mentioned in the sentence. However, in everyday use of plurals like in this example, we only take *finite* pluralities into account, but we are currently concerned with more abstract and theoretical settings in which infinite pluralities need to be considered. We human beings can, in principle, point to or explicitly list finitely many objects, but we cannot do the same for infinitely many.

To plurally refer to infinitely many objects, we usually use definite plural descriptions, such as 'the prime numbers'. However, the truth value of a sentence may rely on the existence of pluralities that cannot be described in a given language, even by impredicative predicates. Let us see one example. For $\mathcal{L}^p_{\mathbb{N}}$-formulae $\varphi(xx)$ and $\theta(x)$, let $\varphi(\{x|\theta(x)\})$ denote the result of replacing each occurrence of $t \prec xx$ in φ with $\theta(t)$ (for each term t). This $\varphi(\{x|\theta(x)\})$ expresses that φ is true of the θs. If $\varphi(\{x|\theta(x)\})$ is true and if $\hat{x}\theta(x)$ is considered a legitimate definite plural noun phrase, then $\exists xx\varphi(xx)$ should be evaluated to be true. However, suppose, in contrast, that $\varphi(\{x|\theta(x)\})$ is false for any plural arithmetical formula θ (even if θ is an impredicative formula). On the one hand, Lévy (1965) showed that it is consistent relative to ZFC that there is a Π^1_2-formula $\Phi(X)$ of second-order arithmetic that admits no projective uniformization; hence, it follows that $\exists X\Phi(X)$ can be true *without* $\Phi(\{x|\Theta(x)\})$ being true for any second-order arithmetical formula Θ. On the other hand, Addison Jr (1959) showed that every second-order arithmetical formula admits a projective uniformization, if $V = L$, and thus $\Phi(\{x|\Theta(x)\})$ is true for some second-order formula Θ whenever $\exists X\Phi(X)$ is true. These results suggest that both $\exists xx\varphi(xx)$ and $\neg\exists xx\varphi(xx)$ may be true when $\varphi(\{x|\theta(x)\})$ is false for any plural

[25] See also footnotes 9 and 17.

arithmetical formula θ; indeed, we can transform Addison Jr's and Lévy's models into models of plural arithmetic in which these are the cases. Hence, both $\exists xx\varphi(xx)$ and $\neg\exists xx\varphi(xx)$ are coherent possibilities, but their truth values rely on the existence (or non-existence) of a plurality that cannot be described by any description.

We are often able to evaluate $\exists xx\varphi(xx)$ to be true by resorting to other objects than those in a given domain of discourse, such as sets of natural numbers. Furthermore, Lévy's (or Addison Jr's) proof provides us with a fairly clear idea of at least one situation (concerning not only natural numbers but also higher-order sets) in which $\varphi(xx)$ is true (false, resp.) for a plurality xx that is coextensive with some set.

However, we are now supposing that pluralities are completely independent of any object outside $\mathbb{N}$, and that all the language-independent factors to determine the truth values of $\mathcal{L}^p_{\mathbb{N}}$-sentences are supposed to be provided by a semantic interpretation of the first-order language $\mathcal{L}_{\mathbb{N}}$ of arithmetic. Hence, $\exists xx\varphi(xx)$ needs to be either true or false even if there is nothing but natural numbers.[26] But, then, what is the fact that is delineated by $\mathfrak{N}$ alone but still determines whether there is some plurality xx such that $\varphi(xx)$? What would a situation be like in which no objects other than the natural numbers exist, yet there exists such an absolutely indescribable, highly complex plurality of natural numbers? I have no idea, and I can hardly imagine that anyone else does.

Our linguistic intuition is of no help in answering these questions. Our ordinary understanding of plurals tells us too little about infinite pluralities that cannot be described within the language we speak—that is, $\mathcal{L}^p_{\mathbb{N}}$, in the case under consideration. Plural primitivists provide a translation of a plural quantifier of PFO into English—that is, $\exists xx\varphi$ is translated into 'there are some things that are φ'—, but this does not add to my understanding of when there are some things that are φ and when there are not. Plural primitivists employ idiosyncratic locutions and say, for instance, that plural quantifiers 'plurally quantify over' the first-order objects, but this informs me of nothing about what infinite plurality can be a value of a plural variable.[27] Our understanding of plurals alone appears to give us no idea of what it is like that there is a plurality, conceived as completely independent of any description or object (set, in particular), that satisfies Lévy's formula.

Taking all this into account, my view is that if pluralities are completely independent of any object outside $\mathbb{N}$, then there is no fact of the matter that determines whether there exists a plurality satisfying Lévy's formula φ. Hence, nothing among those language-independent factors fixed by $\mathfrak{N}$ alone settles whether there is a plurality that satisfies φ. In particular, if pluralities are completely independent of any object outside $\mathbb{N}$, there is no determinate range of plural quantifiers that determines whether $\exists xx\varphi(xx)$ is true. This, in turn, casts a doubt on the definiteness of impredicative $\mathcal{L}^p_{\mathbb{N}}$-formulae in $\mathbb{N}$. Finally, if impredicative $\mathcal{L}^p_{\mathbb{N}}$-formulae are not definite, then the truth of impredicative instances of PCA (at least in $\mathfrak{N}$) is open to doubt. After all, taking pluralities to be completely independent of any objects outside $\mathbb{N}$ would undermine the plausibility of PCA.

26 Furthermore, we can extend Lévy's theorem to higher-order set theory: if there is a transitive model M of ZFC and the existence of a strongly inaccessible cardinal κ, then we can construct a transitive model $N \supset M$ of ZFC such that κ remains strongly inaccessible in N and that there is a second-order *set-theoretic* formula $\Phi(X)$ such that $V_\kappa \models \exists X\Phi(X)$ but $V_\kappa \not\models \Phi(Y)$ for all ordinal definable subsets of V_κ in N. Hence, there is a model of plural set theory in which $\exists xx\varphi(xx)$ is true but $\varphi(yy)$ is false for any yy that is describable in any higher-order set theory.

27 Jané (2005, §10) argues that 'assuming that we understand [plural quantification] well enough for everyday purposes is not a ground for believing that it can support canonical second-order consequence', and I fully agree with him.

It might be objected that while our ordinary understanding of the English word 'set' also tell us little about infinite sets, mathematicians nonetheless have a substantial understanding of them. Mathematician's understanding of infinite sets are largely based on the current set theory as a branch of mathematics, which can hardly be called a theorization of our everyday concept of sets. The (set-based) standard semantics of second-order logic, from which plural primitivists draw an analogy and insight in claiming the semantic determinacy and not-further-committalness of plural sentences, is based on such a mathematical meta-theoretic understanding of sets. Hence, one could try to invent a similarly sophisticated theory of infinite pluralities and base their semantics of plurals on it; Florio and Linnebo's critical plural logic (2020) might be a good candidate for such a theory. Our plural primitivist might thereby object that plural logic under the 'standard' semantics with such a theory of pluralities as the meta-theory fares no worse than second-order logic under the standard semantics.

However, first of all, to me, second-order logic under the standard semantics is not a logic. My argument against plural logic in the last section equally (and more straightforwardly) applies to second-order logic under the standard semantics. It is not neutral to the ontology of sets and is highly dependent on the background set theory; hence, for example, Vänäänen (2001, p. 504) concludes that it is just a 'major fragment of [set theory]', and I believe many logicians and philosophers nowadays share the same view, e.g., Koellner (2010).

Second, regardless of my own view of second-order logic, such a 'sophisticated' theory of pluralities likely requires an axiom equally and similarly controversial as PCA. Second-order logic under the standard semantics renders SCA logically valid because the background set theory postulates the axioms of powerset and separation, the latter of which has the same (or even worse) impredicative character as SCA. Similarly, to make PCA valid, the background theory of pluralities would likely need to postulate axioms of a similar impredicative character; for instance, Florio and Linnebo's (2020; 2021) critical plural logic postulate equally impredicative axioms that correspond to the axioms of powerset and separation.

Third, the new conception of pluralities that would be brought to us by such a 'sophisticated' theory of infinite pluralities might well be quite different from, and foreign to, our everyday conception of plurals. Such a theory and its conception of pluralities might turn out to be no more logical or ontologically innocent than the current set theory and its conception of sets; if so, plural logic would be a 'major fragment of' such a non-logical and/or ontologically committal theory.

My own view is this. The question of what constitutes the range of plural quantifiers is the question of what can be *plurally* referred to; otherwise, the plural primitivist notion of ontologically innocent plural reference would give little support to the ontological innocence of plural quantification. Semantics connects a language and the world, and reference is a semantic relation. The world is independent of languages, and objects exist independently of languages. It is part of the job of a semantic interpretation to supply all the language-independent factors needed to determine the referents and truth values of expressions in the language it interprets. It would be no mystery that the domain of discourse a semantic interpretation specifies contains something to which no noun phrase in the language can refer (under that fixed interpretation). However, if plural expressions are semantically not-further-committal, then all the language-independent facts that are relevant to their referents or truth values must be fully given by a semantic interpretation of the first-order vocabulary, which only contains the information as to what first-order objects exist and of which first-order objects each first-order predicate is true (assuming for simplicity that the vocabulary contains no function or constant symbols). To me, this seems to indicate that there is no fact of the matter, in the circumstance fixed by the semantic interpretation, about plural references that cannot be described by the first-order vocabulary. Hence, in my opinion, while the predicative plural comprehension axiom, in which the condition K in PCA is restricted to be a plural formula with no plural quantifiers, might possibly be a logical axiom (for those who accept the plural vocabulary as logical), the full impredicative PCA cannot.

7. Issues on topic-neutrality

In this section, I give another argument against the logicality of PCA from a different perspective. Namely, I will argue that the assumption of the logicality of PCA has negative consequences upon the topic neutrality of plural logic. My argument in this section does not rely on any strong semantic assumption regarding plural logic. It only requires that the semantics of plural logic is sound for two-sorted first-order logic (where plurals are viewed as the second sort), such as Takeuti's (1987) system $\mathbf{BC}$. Hence, one may work with plural logic under the maximum domain thesis, adopt Florio and Linnebo's plurality-based Henkin semantics, or can regard plural logic as a purely syntactic deductive system.

It is widely thought that logic ought to be *topic-neutral* and *universally applicable* to any topic and subject in the same uniform way. There seems, however, nothing that accommodates and applies to absolutely every theorization of every subject with every conception of the subject matter; if this were a requirement for logic, then even classical logic would fall short of logic because it is incompatible with arithmetic with the intuitionistic conception of natural numbers. In my opinion, topic-neutrality is a matter of degree after all, and I think that whether something is logic ultimately should not be judged solely on the basis of which topics it can cover. Nonetheless, if something claimed to be logic covers too limited a range of topics, this is still a negative sign for its logicality. Therefore, if plural logic is a genuine logic, then it should be topic-neutral to a decent extent.

Restriction of impredicative comprehension abounds in mathematical logic. Typical examples are predicativism; when applied to arithmetic, predicativism results in the Weylian predicative arithmetic or Russelian ramified analysis (or its transfinite extension by Kreisel, Feferman, and Schütte). Finitism gives another example: the second-order system RCA_0^* of arithmetic is defined as $\mathsf{I}\Delta_0(\mathsf{exp})$ plus the second-order axiom of induction and the restriction of SCA to Δ^0_1-formulae: the idea behind the RCA_0^* is that only elementarily recursively definable pluralities are admissible in mathematics from a certain strong finitist point of view.

If PCA is logical, any restriction of SCA is no less 'illogical' under the plural interpretation of second-order quantifiers than the suppression of, say, the identity axioms from first-order logic. Now, the second-order axiom IND of induction,

$$\forall X\big(X0 \wedge \forall n(Xn \to Xn+1) \to \forall n Xn\big),$$

is often considered constitutive of the concept of natural number. For example, Dummett (1994, p. 337) advocates that '[i]t is part of the concept of natural number . . . that induction with respect to any well-defined property is a ground for asserting all natural numbers to have that property'; Lavine (1994, p. 231, n. 24) also contends that 'part of what it is to define a property of natural numbers is to be willing to extend mathematical induction to it'. Whilst both Dummett and Lavine adopts the property interpretation of second-order quantifiers here, the same conclusion applies to IND under any other interpretation on the same ground: that is, any multiplicity of natural numbers in any interpretation of the multiplicity, whether it is understood as a property, set, of plurality of natural numbers, has the least element. Hence, any system of plural arithmetic naturally postulates IND on this ground. However, then, all sub-systems of Z_2 are deprived of serious mathematical and/or foundational significance under the plural interpretation of second-order quantifiers, as 'illogical', which renders much of traditional proof theory insignificant and worthless; thereby, the degree of universal applicability and topic-neutrality of plural logic is considerably reduced.

A possible rejoinder from our plural primitivist might be that the systems investigated in proof theory should be understood not as systems in plural logic but as systems in two-sorted first-order (singular) logic whose subject matter is constituted by natural numbers *and* 'ones over many' over natural numbers of a certain kind, such as sets or (platonist) properties of natural numbers.

However, second-order arithmetic under the plural interpretation and that under, say, the set interpretation share the same first-order part anyway. Hence, if plural logic is a genuine logic, then all the *first-order* consequences of Z_2 are 'logical consequences' of Robinson Arithmetic Q and IND (under the plural interpretation). There has been proposed a plethora of purely first-order theories of arithmetic from various foundational points of view that are proper extensions of Q and (interpretable in) proper sub-systems of Z_2.[28] All these theories only concern the common part of the plural interpretation and the set interpretation of second-order arithmetic, and the difference of the two interpretations is irrelevant to them. However, they all fall far below the first-order part of Z_2. Hence, if plural logic with PCA is a genuine logic, and if arithmetical induction is constitutive of the concept of natural numbers, then these theories are all rendered as 'illogical' and insignificant, which still trivializes much of proof theory. Many plural primitivists are perhaps ready to bite the bullet and accept such a large-scale debunking of the traditional proof theory, but it is surely an unattractive and unpleasing option for many philosophers and mathematicians.[29]

Moreover, the 'two-tiered' approach under consideration is faced with a further difficulty when applied to second-order set theory (also known as class theory). What Z_2 is to arithmetic is what the Morse-Kelley theory MK is to set theory, and the study of subsystems of MK has been rapidly developed in recent years from various foundational and philosophical perspectives.[30] What classes are has been one of the central questions in the philosophy of set theory, and the set interpretation is no longer possible for classes due to the existence of proper classes. One of the major merits of plural primitivism is alleged to be that the plural interpretation offers an answer to this question: namely, quantification over classes is plural quantification over sets. However, to maintain the foundational significance and value of the study of subsystems of

[28] Among such theories are the Kreisel-Feferman-Strahm theories of unfolding (Feferman and Strahm, 2000), the Turing-Feferman theories of transfinite progressions of consistency statements or reflection principles (Feferman, 1962; Turing, 1939), first-order theories of generalized inductive definitions and their variants, Feferman's theories of reflective closures (Feferman, 1991), and various axiomatic theories of truth (see Halbach, 2010).

[29] Hazen (1993) also considers the 'two-tiered' view under consideration here: he pointed out that it puts plural primitivists in 'the anomalous position of holding that someone (the predicativist) who accepts part of second-order logic is ontologically committed to more than someone who accepts all of it!'. Although he seems to share essentially the same worry with me, his contention here (sometimes called 'Hazen's puzzle') is already preempted by the plural primitivist rejoinder under consideration. According to the rejoinder, what predicativists accept is a theory of natural numbers *and* sets, while what plural primitivists accept is a theory purely about natural numbers; the former is not 'part of' the latter.

[30] See (Jäger, 2009), (Jäger and Krähenbühl, 2010), (Fujimoto, 2012, 2023), (Sato, 2014, 2015), (Gitman and Hamkins, 2016), and (Gitman et al., 2020), for example.

MK, the 'two-tiered' approach requires classes to be given a different interpretation than the plural one, whereby plural primitivism loses one of its alleged major merits.

8. Summary

Let me summarize what I have argued. First, PCA cannot be justified as a schema of definitions (§2). Second, PCA is hardly a trivial, *a priori*, self-explanatory truth (§3). Third, the orthodox Gödel-Bernays 'realist' justification of impredicative definitions in set theory cannot be employed in justification of PCA as a logical truth (§4). Fourth, one of the central tenets of plural primitivism, the semantic determinacy and not-further-committalness of plurals, suggests the maximum domain thesis, which makes plural logic with PCA dependent on ontology (§5 and §6). Fifth, PCA severely reduces the topic-neutrality of plural logic from the viewpoint of the current practice of logic. Hence, I conclude that PCA is not a logical axiom.

However, even if my conclusion is accepted, it might be argued that the alleged ontological innocence of PCA would still be a significant merit even as a *non-logical* principle; for example, if Z_2 under the plural interpretation is not ontologically committed to anything beyond natural numbers, then it fares better than Z_2 under the set interpretation, in terms of ontological parsimony. I conclude this article with a brief comment on this issue without in-depth discussion.

Even as a non-logical principle, the truth of PCA must be justified whenever it is postulated. Hence, the question is whether such a justification can be ontologically innocent and less metaphysically laden than a justification for SCA. My argument so far seems to indicate that it cannot. Regardless of whether PCA is taken as a logical axiom, it can be justified neither as a schema of definitions nor as a trivial truth: my arguments against these two types of justifications still stand stand regardless of whether PCA is logical. Realism about pluralities might provide a justification for PCA as a non-logical principle. However, as I argued in §6, without some connection to objects outside the domain of a first-order structure, the truth of PCA is not guaranteed.[31] And if justification of PCA appeals to such a connection to extra objects, then it is no longer ontologically innocent. Thus, even as a non-logical principle, PCA does not appear to fare significantly better than SCA in terms of ontological parsimony.

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[31] Hence, in my view, PCA is false when the first-order domain contains absolutely every object.

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