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The Plural Iterative Conception of Set

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Abstract: Georg Cantor informally distinguished between "consistent" and "inconsistent" multiplicities as those many things that, respectively, can and cannot be thought of as one, i.e., as a set. To clarify this distinction, the recent debate filtered the logic of plurals through two main approaches to the process of set-formation: limitation of size (Burgess) or set-theoretic potentialism (Linnebo). In this paper I propose a third route through the development of a plural iterative conception of set. Inspired by Tim Button's Level Theory, I define and axiomatize the notion of a plural level, which explains Cantor's multiplicities either as level-bound (consistent) or level unbound (inconsistent) pluralities. While this framework is clearly in contrast with the limitation of size view, it also revives a plausible actualist picture prematurely dismissed by the advocates of potentialism.

Keywords: Iterative conception, set theory, plural logic, Cantor, level theory.

1. Introduction

In 1895 Georg Cantor opened one of his articles with the following definition of set:

> By an "aggregate" [*Menge*] we are to understand any collection into a whole [*Zusammenfassung zu einem Ganzen*] M of definite and separate objects m of our intuition or our thought. These objects are called the "elements" of M. In signs we express this thus: $M = \{m\}$. (Cantor, 1895, p. 85)

The reading of $'m'$ as a plural variable, $'mm'$ in modern notation, seems quite natural and has already been extensively defended in the literature.[1]

The idea that sets are obtained by abstraction from a plurality of definite and fixed objects had already been pursued by Cantor in the *Grundlagen*:[2]

[1] See Burgess (2004); Florio and Linnebo (2021); Linnebo (2010, 2013); Oliver and Smiley (2016, 2018).

[2] One could also interpret this definition as opposed to the previous one, in that Cantor seems to be moving from a logical (1883) to a combinatorial (1895) notion of set. In fact, "united into a whole by some law" seems to suggest a conception where sets are pinned down by conditions $'\phi'$ and not just by the plurality of their elements, as it is instead clear from the above definition. However, one could still take "aggregate of determinate elements" to be something like a proto-notion of a plurality. This is even more so in light of the interpretation advanced in the present paper, which endorses the full principle of comprehension for pluralities, bringing them closer to proper classes.

> In general, by a 'manifold' or 'set' I understand every *multiplicity* [jedes Viele] *which can be thought of as one*, i.e. every aggregate [Inbegriff] of *determinate elements* which can be *united* [verbunden] *into a whole by some law*. (Cantor, 1883, p. 916, my emphasis)

The same thought resurfaces also in two famous letters to Hilbert and Dedekind dated, respectively, 1897 and 1899:

> I say of a set that it can be thought of as *finished* [...] if it is possible without contradiction (as can be done with finite sets) to think of *all its elements as existing together* [...]; or (in other words) if it is possible to imagine *the set as actually existing with the totality of its elements*. [...] And so too, in the first article of [Cantor (1895)], I define a 'set' [...] at the very beginning as an 'assembling together' [Zusammenfassung]. But an *'assembling together'* is only possible if an *'existing together'* [Zusammensein] is possible. (Cantor, 1897, pp. 927–928, my emphasis)

> If we start from the notion of a *definite multiplicity* [Vielheit], [...] it is necessary [...] to distinguish two kinds of multiplicities (by this I always mean *definite* multiplicities). For a multiplicity can be such that the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite or inconsistent multiplicities*. [...] If on the other hand the *totality of the elements* of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into '*one thing*', I call it a *consistent multiplicity* or a 'set'. (Cantor, 1899, pp. 931–932, my emphasis)

We can extract two main ideas from these passages. The first is that Cantor conceived *set-formation* as *abstraction from pluralities (i.e., multiplicity, Zusammensein) to sets (i.e., one thing, Zusammenfassung)*:[3] $mm \mapsto \{mm\}$. The second is that not any "*multiplicity*" can form a set, but only those that are "*consistent*". Cantor spells out this notion in various ways and some become more perspicuous if read through the contemporary development of plural logic after George Boolos' seminal work.

Plural logic has been traditionally developed as a form of higher-order (read "second-order") quantification, hence tied to a Principle of Comprehension, which generates pluralities from any given condition ϕ (with xx not free):

$$\exists xx \forall x (x \prec xx \leftrightarrow \phi(x)) \tag{P-COMP}$$

As Linnebo (2010); Yablo (2006) clearly show, a version of Russell's Paradox quickly emerges if we take only Cantor's first idea without the second, namely if we don't restrict our pluralities to "consistent multiplicities", whatever it may mean.[4] That is, if we assume that any plurality whatsoever forms a set, formally

[3] This could also be framed in terms of the conception of "*set-as-one*", to be contrasted with the "*set-as-many*" conception defended by Stanisław Leśniewski in the early days of the discipline (see Potter, 1990, §1.1 and Fraenkel, Bar-Hillel, and Lévy, 1973, §11.1) and clearly articulated already by Russell in *The Principles of Mathematics*, §74 (1903. See also Klement, 2014).

[4] Cantor was well aware of this, although he focused on the plurality of all cardinal numbers, which, if collapsed, leads to the paradox that bears his name:

$$\forall xx \exists x(x = \{xx\}) \qquad \text{(COLLAPSE)}$$

it suffices to plug in the plurality $'rr'$ of all non-self-membered sets, generated by P-COMP, to obtain the antinomic Russellian set.[5] Therefore, coming back to his explanation of "(in-)consistent multiplicity", to formulate a coherent account of sets *qua* obtained from pluralities we need to make Cantor's original idea more precise. While the recent literature clarified these notions either in terms of a limitation of size view (Burgess, 2004; Pollard, 1996) or by outlining a potentialist account of sets (Florio and Linnebo, 2021; Linnebo, 2013), it seems that a gap has been left open: what about the (non modal) Iterative Conception of Set?[6]

The present paper fills this gap and explores a way of clarifying Cantor's distinction through the means offered by the most popular conception of set (Incurvati, 2020). Despite its great popularity, the vast majority of axiomatizations of the Iterative Conception are first-order and overlook the first Cantorian idea of conceiving the process of set construction as a process of plural-to-set abstraction. My aim is to axiomatize a conception of set that sharpens Cantor's ideas through means (the iterative picture) that are not to be attributed directly to him (see Ferreirós, 2007, Epilogue) and that, in the end, sanction the usual axioms of Zermelo-Fraenkel set theory. As Oliver and Smiley observe, "historians agree that an iterative conception of sets is, as doctors say in court, consistent with Cantor's ideas" (Oliver and Smiley, 2016, p. 265), which legitimizes the project of combining the Cantorian view of sets with the Iterative Conception. While Oliver and Smiley (2016, 2018) also advance a theory that goes in this same direction,[7] their aim is to do justice to what they consider a full-fledged "Cantorian orthodoxy" that rejects things such as singleton and empty sets to match their account of pluralities. Therefore, despite a similar appeal to an axiomatization of the Iterative Conception in terms of levels, our respective projects are different and shall not be confused.[8]

The paper is structured as follows. In §2 I introduce the language of plurals and show how it can articulate the process of set formation. In §3 I introduce the raw idea of the Iterative Conception of Set and explain how it can be integrated with the idea that sets are obtained from pluralities. §4 and §5 are the core of the paper, where I propose an axiomatization of the conception under the label of Plural Level Theory (PLT) and where I introduce its most salient results. In §6 I compare PLT with alternative approaches to the development of a Cantorian view of sets. §7 closes the paper with a summary of the main results. In Appendix A I gather the most technical details of the framework.

2. From Pluralities to Sets

Starting from the overall idea of grounding set theory into plural talk, I proceed to define a plural language and see how it interacts with sets.

> In contrast, infinite sets such that the *totality* of their elements cannot be thought of as 'existing together' [...], and that therefore also *in this totality* are absolutely not an object of further *mathematical* contemplation, I call *'absolutely infinite sets'* and to them belongs the 'set of all alephs'. (Cantor, 1897, pp. 927–928, his emphasis)

[5] To be absolutely precise one would also need a principle of set-abstraction that grants that identical sets are mapped to identical pluralities, formally $'\{xx\} = \{yy\} \leftrightarrow xx \approx yy'$ (see below for the notation). In fact, in accounts like Florio and Linnebo (2021) this principle, being more fundamental, substitutes COLLAPSE from Linnebo (2010).

[6] Whether or not potentialism is an instance of the Iterative Conception is up for debate: Linnebo (2013); Studd (2013) answer positively, while Roberts (MS) frames it as an alternative both to the conception and to Limitation of Size.

[7] Another theory combining the Iterative Conception with plural quantification is presented by Pollard (1985). However, his work is radically different both from mine and from Oliver and Smiley's since he starts from the ordinals as urelements and investigates how second-order quantification *à la* Boolos can make sense of various approaches to the conception.

[8] If one wants to proceed with a comparison, in §6.3. I argue that my approach fares better also if framed within their "Cantorian orthodoxy".

2.1. Pluralities

$\mathcal{L}_\prec$ is an extension of first order logic (FOL) with identity that, along with plural variables $'xx, yy, zz, \ldots'$ and quantifiers, introduces a plural "membership" predicate $'\prec'$. This takes singular terms on the left and plurals on the right, so $'x \prec xx'$ is to be read as $'x$ is among/one of the xx'. $\mathcal{L}_\prec$ adds two axioms to FOL:

$$\exists xx \forall x(x \prec xx \leftrightarrow \phi(x)) \tag{P-COMP}$$

$$\forall xx, yy(\forall u(u \prec xx \leftrightarrow u \prec yy) \rightarrow (\Phi(xx) \leftrightarrow \Phi(yy))) \tag{P-INDISC}$$

The first is the principle P-COMP from above, which grants an unrestricted generation of "*definite*" pluralities, both "consistent" and "inconsistent". For me this is a crucial part of the Cantorian spirit of the project since Cantor seems to be explicitly committed to a free generation of "definite multiplicities", which I read as "*pluralities defined by a comprehension principle*". The crucial point is then demarcate between those definite multiplicities that can and those that cannot collapse into sets. Nonetheless, the acceptance of P-COMP, is a contentious point between my proposal and potentialist accounts of sets, both modal and non-modal (see §6.2.). The second axiom is just indiscernibility of identicals for pluralities.[9] My formulation follows Burgess (2004) and it is enough to preserve the extensional nature of pluralities in the sense that co-extensionality (left-hand-side) implies indiscernibility (right-hand-side).[10]

We start with the definition of the derived relation of (proper) *plural inclusion* (or *subplurality*) and *plural identity* as in Florio and Linnebo (2021, §2.3):[11]

Definition 2.1 (P-IND). [12]

(i) $bb \preccurlyeq aa \leftrightarrow_{def} (\forall x \prec bb)x \prec aa$
(ii) $bb \precnsim aa \leftrightarrow_{def} (\forall x \prec bb)x \prec aa \wedge \neg(aa \preccurlyeq bb)$
(iii) $bb \approx aa \leftrightarrow_{def} bb \preccurlyeq aa \wedge aa \preccurlyeq bb$

Just as I differentiate between plural and singular identity with $'\approx'$ and $'='$, in the same way, I use $':\approx'$ in definitions of specific pluralities in place of $':='$.

[9] Florio and Linnebo (2021, p.19) use lowercase schematic letters specifying that $'\phi(xx)'$ stands for the standard substitution of all free occurrences of some free variable vv by xx whenever vv can be substituted by xx in ϕ. Here I prefer to adopt capital letters to avoid the following misunderstanding, which one can easily imagine arising if one is not already familiar with the language of plurals: one may interpret $\phi(xx)$ as being the plurality of all x such that $\phi(x)$ characterized by P-COMP above. Due to pluralities being extensional entities (as can be easily derived from P-INDISC), there is just one such plurality (as in the case of sets) and, under this interpretation, it would make no sense to speak of "all pluralities that are $\phi(xx)$" as done in the axioms below. To avoid such misunderstanding, I adopt the present notation to tag schematically collective predicates.

[10] Alternatively, one could add a primitive plural-identity predicate $'\approx'$ with the axioms: P-COMP +

$$\forall xx, yy(xx \approx yy \rightarrow (\Phi(xx) \leftrightarrow \Phi(yy))) \tag{P-INDISC$^\approx$}$$

$$\forall aa \forall bb(\forall x(x \prec aa \leftrightarrow x \prec bb) \rightarrow aa \approx bb) \tag{P-EXT}$$

This is similar to axiomatic set theory where Indiscernibility is a logical axiom and implies the right-to-left direction of Extensionality. Here we can dispense $'\approx'$ as a further primitive and define it instead, while P-INDISC is already apt to do the job of Extensionality in granting that the notions defined below pin down pluralities in a unique way. Moreover, given Def. 2.1 below, we can easily derive P-INDISC$^\approx$. For this reason I go with the option more customary in the literature on plurals (Burgess, 2004; Florio and Linnebo, 2021; Oliver and Smiley, 2016). For more on plural primitives see Linnebo (2007, §3).

[11] They introduce these definitions before introducing P-INDISC, but the order should be reversed.

[12] For readability, I adopt the same conventions as Button (2021, §0). That is, I concatenate infix conjunctions and I abbreviate bounded quantifiers in the usual way. E.g., I write $'(\forall x \prec yy)\phi'$ for $'\forall x(x \prec yy \rightarrow \phi)'$ and $'(\forall x : \Psi)\phi'$ for $'\forall x(\Psi(x) \rightarrow \phi)'$. The rest is the same for the existential quantifiers and any predicate $'\Psi'$ or infix predicates other than $'\prec'$.

2.2. Sets

Coming to the first Cantorian idea of conceiving sets as collapsed pluralities, I enrich $\mathcal{L}_{\prec}$ with an additional primitive non-logical symbol '$\ltimes$' of the reverse type of '$\prec$', where '$xx \ltimes x$' is to be read as "the plurality xx collapses into the set x". I denote the new language as $\mathcal{L}_{\prec,\ltimes} = \{\ltimes\}$. This allows set-membership to be defined:[13]

Definition 2.2 (P-IND)**.** $x \in a \leftrightarrow_{def} (\exists aa \ltimes a)x \prec aa$, alternatively, $(\forall aa \ltimes a)x \prec aa$

To avoid making the notation too cumbersome from now on I will freely use '$\in$' and its derived relations '$\subseteq / \subset$' thoroughly. However, bear in mind Def. 2.2 as making explicit Cantor's first idea concerning the process of set-formation.

The behavior of the collapse relation is further prescribed to be extensional:[14]

$$\forall aa, bb \forall a, b((aa \ltimes a \wedge bb \ltimes b) \to (a = b \leftrightarrow aa \approx bb)) \qquad (\text{EXT}_p)$$

An obvious consequence of EXT_p is that $(xx \ltimes a \wedge xx \ltimes b) \to a = b$, that is, if it exists, the collapse of a plurality into a set is uniquely determined (and vice-versa). This allows the introduction of functional notation '$\uparrow xx / \downarrow x$' to directly refer, respectively, to the collapse/uncollapse of a plurality/set as terms.[15] To differentiate between plural and set abstraction, I use the double pipe to tag the former, that is, I denote the set a of all the ϕ as '$a = \{x : \phi(x)\}$' and the respective plurality aa as '$aa \approx \|x : \phi(x)\|$'.[16] Combined together these yield other useful definitions:[17]

Definition 2.3 (P-INDISC, EXT_p)**.**

(i) $\uparrow aa := \{x : x \prec aa\}$
(ii) $\downarrow a :\approx \|x : x \in a\|$
(iii) $a \cap aa :\approx \|x : x \in a \wedge x \prec aa\|$
(iv) $a \sqsubseteq aa \leftrightarrow_{def} (\forall x \in a)x \prec aa$
(v) $a \sqsupseteq aa \leftrightarrow_{def} (\forall x \in aa)x \in a$
(vi) $a \blacktriangleright aa \leftrightarrow_{def} (\exists c \prec aa)a \subseteq c$

Concerning (1) and (2) note that, while $\downarrow a$ always exists if a does by Def. 2.2, the same is not true for $\uparrow aa$ and aa since this depends on the sharpening of "consistent multiplicity" we are after. In this sense, while '$\downarrow$' represents a total function, '$\uparrow$' stands for a partial function that sometimes is non-denoting. The aim of our theory can then be restated as singling out when '$\uparrow$' behaves like a total function. This is a further difference with system and Oliver and Smiley (2016), who simply accept the partiality of the 'set of' function (see §6.3. below).

Therefore, concerning Def. 2.3(1), whenever we have the collapse of a plurality into a set, but we miss the qualification of "consistency", we shall read that as: *if the collapsed set exists*, then it

[13]And also of a set-predicate, where needed: $ß(a) \leftrightarrow_{def} (\exists aa \ltimes a)$. This is the same as the axiom HEREDITY (3.1) from Burgess (2004, p. 198), who takes 'ß' as a primitive. I can define it because I am not appealing to a principle of reflection, as Burgess does, and so I am not worried about relativized formulae. Nonetheless, the definition of 'ß' has the same effect of HEREDITY, hence its name: all the (set)elements of a set are included in the universe of discourse as soon as the set is introduced

[14]This axiom is also present in Burgess (2004) as 3.2. However, since Burgess assumes a set-predicate as a further primitive (see fn. 13), he is forced to assume (or, better, derive) a principle of PURITY (7.2) to derive the usual set-theoretic axiom. On the contrary, here the set-theoretic axiom follows from P-INDISC, EXT_p and Def. 2.2, which already excludes ur-elements from the scope of '$\ltimes$'.

[15]This is also used in the theory of (linguistic) groups after Landman (1989).The function $\uparrow$ (written '$\{\}$') is also used by Oliver and Smiley (2016, §14.7). However, they assume it as a primitive and say "We do not need extensionality as an axiom, however, since it is implicit in the syntactic classification of $\{\}$ as a function sign" (p. 268). On my end, starting from a relational symbol like '$\ltimes$' and justify the introduction of functional notation on the basis of its extensional behavior makes things more perspicuous and allows to avoid some of the oddities of their system (see §6.3.). Overall, unless one believes that all pluralities collapse into sets, like Florio and Linnebo (2021), choosing a relational collapse predicate over a functional one seems the best choice.

[16]While pluralities can be defined as any collection of the form above due to P-COMP, for obvious reasons sets cannot. Following textbooks presentations, the notation above generally defines a class, which is a set if it exists. See Kunen (2013).

[17]Some already in Burgess (2004) and Oliver and Smiley (2016).

is uniquely determined by the pluralities it is collapsed from. For instance, if the following '$\uparrow x$' exist, we have these equivalences:

$$\uparrow \|x:\phi(x)\| = \{x:\phi(x)\} \qquad \downarrow \{x:\phi(x)\} \approx \|x:\phi(x)\|$$
$$\uparrow\downarrow a = a \qquad \downarrow\uparrow aa \approx aa$$

Furthermore, 2.3(3) abuses notation from set-intersection to define the intersection between a set and a plurality. I take this to be uncontroversial as they are both collections subject to some kind of membership: since something can be both a member of a set and one among a certain plurality, there should be a collection gathering together what a set and a plural have in common. I take this to be a plurality because of the "*conceptual priority*" that characterizes pluralities with respect to sets in the Cantorian picture:[18] one could collapse a set-plural intersection to obtain the corresponding set: $\uparrow (a \cap aa) \coloneqq \{x : x \in a \wedge x \prec aa\}$. I also take intersections and unions between pluralities (for which I also abuse notation) to be uncontroversially defined just like their set-theoretic equivalents:

$$\bigcap xx \approx \downarrow \bigcap \uparrow xx \qquad \bigcap x = \uparrow \bigcap \downarrow x \qquad \bigcup xx \approx \downarrow \bigcup \uparrow xx \qquad \bigcup x = \uparrow \bigcup \downarrow x$$

Finally I also define the notation for the following special items:

Definition 2.4 (P-INDISC, EXT$_p$).

(i) $ee :\approx \|x : x \neq x\|$ (i.e., the empty plurality)
(ii) $\uparrow ee = \emptyset$ and $\downarrow \emptyset \approx ee$
(iii) $\emptyset\emptyset :\approx \|x : x = \emptyset\|$ (i.e, the singleton plurality of the empty set)
(iv) $\uparrow \emptyset\emptyset = \{\emptyset\}$ and $\downarrow \{\emptyset\} \approx \emptyset\emptyset$

As mentioned above, the empty plurality has a somewhat controversial status (Oliver and Smiley, 2016). Here I take its existence to be uncontroversially granted simply by P-COMP. The reason is that our primary focus is the development of a conception of sets, so I am insensitive to whether or not the empty plurality has a correspondence in the natural language.[19] Similar considerations should dissolve also worries related to singleton pluralities. In any case, those worried by these pluralities may refer to the aforementioned Oliver and Smiley (2018) for a parallel project of a Cantorian "theory" of sets or may try to adapt this project of a "conception" of sets to Button (2021, Appendix A and B), who develops a level theory with urelements.

3. Iterations

Now that we have an idea of how pluralities and sets interact between each other and of how the first can, in principle, be collapsed into the second, we shall provide a structure to situate this intuition which is also our solution to the quest for consistent multiplicities. Then, we shall sketch a preliminary picture of how the process of plural-to-set abstraction can be placed within this iterative framework.

[18] Whether this priority can be turned into something like metaphysical dependence or some other hyperintensional notion is something I remain neutral on in the spirit of Incurvati's minimalism concerning the Iterative Conception (Incurvati, 2012, 2025).
[19] A more pressing issue concerning the empty plurality would concern its interaction with the "nothing over and above" conception advocated by Roberts (2022). I leave this discussion to further more philosophical work.

3.1. Iterating Sets

The structure is provided by the Iterative Conception of Set. Described in the most general way possible, this is a way of introducing the universe of sets in a more explicit and structural manner[20] and in more constructional terms,[21] that is, by directly regimenting a pre-theoretic story of how sets are generated through an iterated set-construction procedure:

> **Iterative Conception of Set**: sets are generated in layers. Each set lives at some layer. At every layer some set-generating operation outputs new sets by using as inputs sets found at previous layers. There are no further sets outside those generated in this way.

The operational talk in terms of generation follows an approach first presented by Forster (2008) and thoroughly developed by Button (2024) which places at the heart of the conception just the idea of *iterating some constructional procedure*. This contrasts the widespread impression that intrinsically ties the conception to the cumulative hierarchy and its set-generator: power-set. To use Forster words: “there is nothing in the idea of sets as conceived *iteratively* that says that there should be only one constructor [i.e., power-set]” (Forster, 2008, p. 99, emphasis in the text).[22]

The present project is not far from the original idea that the Iterative Conception is an explicit (i.e., non-recursive) way of describing and axiomatizing the cumulative hierarchy (see Montague, Scott, and Tarski, unpublished). The original shift is that it develops the conception as a way of reconciling (and making sense of) Cantor’s original ideas concerning the process of set-formation with(in) the most popular development of the structure of the universe of set. In the end, the output resembles the hierarchy of sets under most respects, with the crucial twist that the set-generating operation is *plurality-to-set abstraction*, namely ’$\ltimes$’, following Forster, in focusing on a different constructional procedure.[23] This twofold nature is also justified by the fact that a Cantorian conception of sets as abstracted from pluralities is also (implicitly) present in authors explicitly engaged with the Iterative Conception. Here’s how Dana Scott, introduces one the first axiomatizations of the conception: “But note that our original intuition of set is based on the idea of having *collections of already fixed objects*” (Scott, 1974, p. 207, my emphasis). Just as for Cantor’s original remarks, it is most natural to interpret the “already fixed objects” as a plurality. The same idea is also present in George Boolos’ famous paper which popularized the conception among philosophers:

> For when one is told that a set is a collection into a whole of definite elements of our thought, one thinks: Here are some things. Now we bind them up into a whole. *Now* we have a set. We don’t suppose that what we come up with after combining

[20]The alternative approach, the one originally championed by Zermelo (1930), is the axiomatic one: start with some principles that describe the behavior of set and then simply generate the cumulative hierarchy (i.e., V) via a definition by transfinite recursion.

[21]Be careful not to confuse “constructional” with “constructive”. The first term is borrowed from the literature on constructional ontologies and indicates a step-by-step construction of entities (sets in this case) through an iterative process. The second refers to Gödel’s constructible universe, or L, which is a way of conceiving the Iterative Conception in predicative-friendly terms, but that is not the focus of this paper (see fn. 22).

[22]Furthermore, I remained silent on whether, at each layer, one finds all possible outputs the set-generator. This is in line with the most recent literature on the weak conception (Barton, 2024) that detaches the Iterative Conception from the cumulative hierarchy under another respect, namely that related to the *maximality* of the generative process. In other terms, besides there being more than just one constructor, there are also more ways to use each constructor, depending on whether one wishes to obtain all possible results of the application of that constructor at the very next layer. The most straightforward example is the one that compares V and L: the constructor is always power-set, but in the second case its strength is considerably weakened by yielding only definable subsets.

[23]I still move within a strong notion of the conception (see fn. 22) and interpret the process of set-formation as an abstraction from all possible pluralities found at some stage to all possible sets obtained from those pluralities.

> some elements into a whole could have been one of the very things we combined. (Boolos, 1971, p. 18, emphasis in the text)

In particular, the lasso metaphor that he attributes to Kripke in a footnote can be taken as a natural description of the process of plural-to-set abstraction: *forming a set is like throwing a lasso around some things, but you cannot throw a lasso if the things you are trying to group are not definite and fixed, on pain of paradox*. This is largely reminiscent of Cantor's talk of consistent and inconsistent multiplicities, which makes the decision of clarifying these notions within the conception rather natural.

3.2. Iterating Pluralities of Sets

A first suggestion on how the Iterative Conception can sharpen the concept of "consistent multiplicity" can already be found in one of its seminal formulations:

> This concept of set, however, according to which *a set is something obtainable from the integers (or some other well-defined objects) by iterated application of the operation "set of"*, not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naive" and uncritical working with this concept of set has so far proved completely self-consistent. (Gödel, 1947, pp. 518-519, my emphasis)

Here Gödel seems to be suggesting two things. The first is a rather straightforward and undisputed fact regarding the conception, namely its being a *combinatorial* rather than a *logical* view of the process of set-formation.[24] The second is that this combinatorial aspect can simply be instantiated by iterated application of a "set of" operation to some "well-defined objects".

Although the identification between this operation and my '$\ltimes$' may seem straightforward, we need a caveat first. As a matter of fact, Gödel (1951, fn. 5) specifies that by "set of" he means power-set, tying the Iterative Conception to this privileged constructor and its output (i.e., the universe V) as described by Forster in the previous passage. However, we should not feel discouraged by this. Our purpose is to give a precise definition of the notion of consistent multiplicity advanced by Cantor and to do so we are appealing to the tools offered by the Iterative Conception. It is then legitimate to adapt these tools to the present context, namely one where sets are generated by abstraction from pluralities. Moreover, pluralities perfectly capture the combinatorial aspect embodied by the Iterative Conception where sets are pinned down by their elements rather than an application condition.[25]

Therefore, as a first approximation, we can define as "consistent" all those multiplicities that are part of an iterative process of set-generation starting from a "safe" multiplicity, like the integers, or an empty multiplicity (i.e., ee). In what follows I shall show how this idea, in its latter instance,[26] can be sharpened further through a process that parallels the historical development of the Iterative Conception.

[24] See Incurvati (2020); Maddy (1983); Parsons (1974).

[25] See Florio and Linnebo (2021); Linnebo (2010).

[26] The former instance has been advocated by Oliver and Smiley (2016) in their attack on empty and singleton sets and defense of ur-elements in the context of the conception. Although I agree that the most faithful interpretation of Gödel's remarks would be taking the integers as an ur-element basis, it is also true that, as I noted above, Gödel also had in mind the iteration of a straightforward set-theoretic operation like power-set. For this reason, I think that if it is legitimate to distance myself from Gödel by iterating plural-to-set abstraction (as Oliver and Smiley did), then it is also legitimate to further distance myself by retrieving the pure universe of sets. Furthermore, it can also be argued that this is a fake distancing since Gödel showed no issues with the pure universe

3.3. *The Plural Hierarchy*

As a starting point, let's adapt Gödel's "set of" operation to the present setting where sets are generated from pluralities. We can move from the standard notion of power-set to a plural-based one, through the following operation, which pins down the sets generated from the subpluralities of a given plurality:

Definition 3.1 (P-IND)**.** $x \trianglerighteq aa \leftrightarrow_{def} (\exists xx \preccurlyeq aa) xx \ltimes x$

This yields the "power-plurality" of a plurality 'xx' as the plurality of all the sets obtained by collapsing all the sub-pluralities of xx

Definition 3.2 (POWER PLURALITY)**.** $\mathcal{S}(xx) :\approx \|x : x \trianglerighteq xx\|$

The equivalences deriving from Def. 2.3 yield a straightforward connection between this notion and the standard set-theoretic power-set:

Fact 3.3. For any x and xx: $\mathcal{S}(xx) \approx\downarrow \mathcal{P}(\uparrow xx)$ and $\mathcal{P}(x) =\uparrow \mathcal{S}(\downarrow x)$

For the sake of argument, let's put this definition at work in a context where ordinal indexing and transfinite recursion are primitively available, as in Pollard (1985), to obtain an approximate and preliminary sharpening of Gödel's remark. This serves the heuristic purpose of offering an intuitive grasp of the final picture, which will be perfected later, once we define our theory of levels.

Definition 3.4 (vv_α)**.** $vv_0 \approx ee; \quad vv_{\alpha+1} \approx \mathcal{S}(vv_\alpha); \quad vv_\lambda \approx \bigcup_{\alpha<\lambda}(vv_\alpha), \ (\lambda \text{ limit})$

That is, iterating a plurally-interpreted "set-of" operation along the same process described by Gödel leads to a cumulative hierarchy of pluralities. This idea can be made more explicit if read in connection to the standard set-theoretic V_αs, again straightforward from Def. 2.3:

Fact 3.5. $vv_\alpha \approx\downarrow V_\alpha$ and $V_\alpha =\uparrow vv_\alpha$.

In other terms, we have defined a hierarchy of pluralities where each layer collapses into the corresponding one of the hierarchy of set, and vice-versa for the uncollapse. Imagine the two hierarchies sandwiched between one another, alternating a plural and a set-theoretic layer one at a time starting from the empty plurality.[27]

of sets, but instead was one of its most vocal advocates (see Kanamori, 2004). Therefore, I think that, rather than telling against my project, an appeal to Gödel tells against starting with ur-elements instead.

[27] This image can be made more perspicuous through a redefinition of both hierarchies in terms of the other, which can go in multiple ways:

$$
\begin{array}{lll}
vv_0 \approx\downarrow V_0; & vv_{\alpha+1} \approx \mathcal{S}(\downarrow V_\alpha); & vv_\lambda \approx \bigcup_{\alpha<\lambda} (\downarrow V_\alpha), \ (\lambda \text{ limit}) \\
V_0 =\uparrow vv_0; & V_{\alpha+1} = \mathcal{P}(\uparrow vv_\alpha); & V_\lambda = \bigcup_{\alpha<\lambda} (\uparrow vv_\alpha), \ (\lambda \text{ limit}) \\
vv_0 \approx\downarrow V_0; & vv_{\alpha+1} \approx\downarrow \mathcal{P}(V_\alpha); & vv_\lambda \approx\downarrow \bigcup_{\alpha<\lambda} (V_\alpha), \ (\lambda \text{ limit}) \\
V_0 =\uparrow vv_0; & V_{\alpha+1} =\uparrow \mathcal{S}(vv_\alpha); & V_\lambda =\uparrow \bigcup_{\alpha<\lambda} (vv_\alpha), \ (\lambda \text{ limit}) \\
vv_0 \approx ee; & vv_{\alpha+1} \approx\downarrow \mathcal{P}(\uparrow vv_\alpha); & vv_\lambda \approx\downarrow \bigcup_{\alpha<\lambda} (\uparrow vv_\alpha), \ (\lambda \text{ limit}) \\
V_0 = \emptyset; & V_{\alpha+1} =\uparrow \mathcal{S}(\downarrow V_\alpha); & V_\lambda =\uparrow \bigcup_{\alpha<\lambda} (\downarrow V_\alpha), \ (\lambda \text{ limit})
\end{array}
$$

Alternating between up and down arrows should give an idea of how one can move between plural and set layers.

This way of conceiving the process of set construction basically condensates what we could label the "pre-history" of the Iterative Conception, namely the thirty years that went from the discovery of the paradoxes to Zermelo's formulation of the Cumulative Hierarchy. If we indulge for a moment in the practice of counterfactual history, it makes sense to imagine that the early set theorists would have ended up with the vv_α had they given more space to Cantor's intuition that the process of set formation is a process of abstraction from pluralities. Just like the set-theoretic hierarchy grounds a paradox free (i.e., consistent) process of set formation, we can take the above hierarchy of pluralities to ground a notion of "consistent multiplicity" in the same way. A *consistent multiplicity is any plurality that occurs at some level of the cumulative hierarchy of plurals*, that is:

Definition 3.6 (CONSISTENT MULTIPLICITY). $\mathfrak{C}(xx) \leftrightarrow_{def} \exists\alpha(xx \preccurlyeq vv_\alpha)$

The Cantorian nature of this definition rests also on the fact that there is still room for inconsistent multiplicities. That is, we are not endorsing something like the following principle: $\forall xx\exists\alpha(xx \preccurlyeq vv_\alpha)$. This would be straight away inconsistent with P-COMP since pluralities like the pluralities of all sets do not appear at some level of the plural hierarchy. This principle can be accepted if we restrict comprehension. In fact, Florio and Linnebo (2021) commit to an equivalent axiom formulated in terms of an induction scheme when they prepare the setting for their Critical Plural Logic to derive ZF. I, on the other hand, am committed to a weaker principle, which is sufficient for the purposes of the present theory: $\forall xx(\mathfrak{C}(xx) \rightarrow \exists\alpha(xx \preccurlyeq vv_\alpha))$, i.e., *every consistent multiplicity appears at some level of the plural hierarchy*. This is just a consequence of Def. 3.6 and flags the fact that *the plural hierarchy pins down exactly the consistent multiplicities*, which is a way of sharpening Cantor's claim via Gödel's observation. An analogous formulation, which is a crucial theorem of the theory presented in next section, is that the plural hierarchy is well-founded.

That said, however, the cumulative hierarchy is not all there is to the Iterative Conception. On the contrary, the conception as we know it resulted from the hierarchy once a "notable inversion" has been enacted: "In a *notable inversion*, what has come to be regarded as the underlying iterative conception became a heuristic for motivating the axioms of set theory generally" (Kanamori, 2004, p. 521, my emphasis). That is, once defined via the means of axiomatic set theory, the hierarchy motivates a notion of layer that can be explicitly and non-recursively defined, quantified over and axiomatized. This is, in a nutshell, the inversion brought to the table by axiomatic approaches to the conception. Therefore, in a parallel move, I now proceed to offer an analogous regimentation of what the Iterative Conception looks like if axiomatized against the background of the vv_α. Once again, we can leverage on the work already done in the set-theoretic case in the past 90 years, which leads to the formulation of a theory of plural levels.

4. Plural Level Theory

Plural Level Theory (PLT), as an axiomatization of the Iterative Conception, is a plural reworking of the Level Theory (LT) advanced by Button (2021).

4.1. Intuitive Formulation

Let's start with a pre-theoretic story that captures its general idea:

The Plural Story: sets are generated in layers. *Some things are a layer* just in case they are *all possible sets* of sets found at previous layers. Each set lives at some layer. There are no further sets outside those generated in this way.

PLT takes the plural talk of the story at face value and makes it explicit through an axiomatization. That is, it interprets "all possible sets of sets" as a plurality of sets and regiments this notion of plural layer through an explicit definition.[28]

Before proceeding with the formalization, it is preferable to sketch an informal characterization of how I want the hierarchy of plural levels to behave. In this way we can recognize a pattern that should then lead to a suitable regimentation. The strategy is to parallel the Button-Potter approach of alternating between *histories* (accumulations of levels) and *levels*. Starting from the empty set,[29] we generate the first level, accumulate it in the next history, generate the second level, accumulate all the previous levels in another history, generate a level and so on. As stated in Button and Walsh (2018, 8c), we can generally portray levels as the V_αs and histories as sets of the following form: $\{V_\gamma : \alpha < \gamma\}$.[30]

On our end, representing plural levels as the unbracketing of the set-theoretic levels, that is, our vv_αs is rather straightforward. However, two strategies are now available for what concerns the notion of a plural history, depending on whether we lean more towards being faithful to our set-theoretic aims or towards being "plural-purists" so to speak. Here I focus on the first case, and I take both levels and histories to be the "unbracketing" of Button's set-theoretic notions. Therefore, combining the two heuristics above, plural histories will simply be the following pluralities: $\| \uparrow vv_\gamma : \gamma < \alpha \|$. Here plural histories gather the $\uparrow vv_\alpha$s (i.e., the V_αs), into pluralities. In the second case, plural levels are accumulated into histories while remaining pluralities without being collapsed. To do so it seems that we are forced to appeal to super-pluralities because, being an entity of one order higher than pluralities, they can keep track of the difference between, for instance, vv_0, vv_1 and vv_2.[31] However, since this a rather controversial topic that deserves its own discussion,[32] I leave the formulation of a Super-plural Level Theory to a separate work.

[28]Those familiar with the axiomatizations of the conception may have noted that I intentionally skip one passage from Button's blueprint, namely the formulation of a theory of (plural) *stages* before that of (plural) *levels*. After Boolos (1971, 1989); Kreisel (1965), these are *sui generis* entities primitively quantified over, which seems the most natural first move when axiomatizing the pre-theoretical story. Although parting with the level-theoretic tradition (Montague, Scott, and Tarski, unpublished; Potter, 1990, 2004), Button still has to face the approach popularized by Boolos. He first develops a stage theory and then shows that it makes the same set-theoretic claims of LT, arguing for the latter through an economy-of-primitives argument. Here I can avoid the confrontation with a hypothetic plural-stage-theoretic approach simply because there is none. Even if there was one, this would be an over-complication with no significant advantages or justifications since bare plural quantification because already is a natural way to formalize the plural story. I also subscribe to the reasons advanced by Button in moving from stages to levels: just like he aims for a purely set-theoretical foundation, here I aim for a "purely plural" development of the conception. Moreover, given the tight connection between PLT and LT (see below), I take my case to be analogous in the sense that, if developed, a plural stage theory could easily be made set-theoretically equivalent to PLT.

[29]Which is trivially a history, see Potter (2004).

[30] Note that this equivalence is not just a useful heurstic to explain what levels and histories are in terms of something familiar, but an actual result proved by Button and Walsh. That is, one can prove that the levels are exactly the V_αs and vice-versa.

[31]Take the accumulation of vv_0, vv_1 and vv_2 in the plural hierarchy outlined in the previous section. If we take this to be just a plurality we would have the following result, which loses track of two of the three levels: $\text{HIST}_3 \approx vv_0 + vv_1 + vv_2 \approx ee + \emptyset\emptyset + \emptyset, \{\emptyset\} \approx \emptyset, \{\emptyset\}$. In other terms, since plurals are nothing over and above their members (Roberts, 2022), accumulating those three pluralities together would amount to have a plurality where the empty plurality disappears since it does not contain anything, while the empty set is counted twice and, just as in the case of sets, we don't count the members of a plurality more than once. As a result, in this case, we lose track of vv_0 and vv_1 in the history, hence the appeal to super-pluralities.

[32]See Florio and Linnebo (2021, ch. 9), Linnebo and Nicolas (2008); Nicolas and Payton (2025); Payton (2025).

4.2. Formal Regimentation

Now that we have an intuitive representation of the picture advanced by PLT, we can move to the definition of the relevant concepts. Let's start from the definition of an operation that Button (2021), following Montague, Scott, and Tarski (unpublished); Scott (1974), labels *potentiation*:[33]

Definition 4.1 (PLURAL POTENTIATION)**.** For any aa, let aa's plural potentiation be $\P\P(aa) :\approx \|x : x \blacktriangleright aa\|$.[34]

In other terms, a plural potentiation of a plurality aa is a plurality consisting of all the subsets of members of aa. This mirrors Button's potentiation as the set of all the subsets of the elements of the given set, that is, a super-transitive closure (closure under subset, i.e., $\P(a) := \{x : x \rhd a\}$). So the plural definition coincides with that of a super-transitive plurality. Moreover, it also preserves the "conceptual connection" with power-set observed by Button (2021, p. 439): if the singular potentiation of a singleton set is equivalent to the *power-set* of its unique member ($\P(\{a\}) = \mathcal{P}(a)$), the plural potentiation of a singleton plurality corresponds to the *power-plurality* of its sole member, i.e., the plurality that then collapses in the power-set of its member.

Fact 4.2. If aa is the singleton plurality of the set a, then, if it exists, $\uparrow \P\P(aa) = \mathcal{P}(a)$.

Moreover, since we defined the a new operation, this correspondence can be also tracked town in terms of a power plurality:

Fact 4.3. If aa is the singleton plurality of the set a, then $\P\P(aa) = \mathcal{S}(\downarrow a)$.

Next, we define a *plural history*:

Definition 4.4 (PLURAL HISTORY)**.** $\textsc{Hist}(uu) \leftrightarrow_{def} \forall x \prec uu(x =\uparrow \P\P(x \cap uu))$

Here the definition twists Button's, whose histories are sets whose members are the potentiation of the intersection between the set and their members. The reason is the plural setting: since our aim is to build a plural hierarchy of sets, we need a place to introduce new sets, that is, to tap the "wand" '$\uparrow$' (or '$\ltimes$') to use Forster's and Button's terminology, otherwise we would move in a tiny circle and the iterative process would never kick in. For this reason, a plural history is a plurality whose members are sets collapsed from the plural potentiation of the plural-set intersection between the history itself and its members.

Finally, we define the core concept of a *plural level*:

Definition 4.5 (PLURAL LEVEL)**.** $\textsc{Lev}(ss) \leftrightarrow_{def} \exists uu(\textsc{Hist}(uu) \wedge ss \approx \P\P(uu))$

In other terms, just as in LT, levels are (plural) potentiations of histories, which are initial sequences of levels.

Lastly, we need an additional notion with respect to Button's LT:

Definition 4.6 (BOUNDED LEVEL)**.** $\textsc{Lev}_\beta(ss) \leftrightarrow_{def} \textsc{Lev}(ss) \wedge (\exists tt : \textsc{Lev}) ss \not\preccurlyeq tt$

The reason for this is that, as we shall see in §5.2., the theory sees the first-order domain as a plural level and thus P-COLLAPSE below stated for levels in general would force us to collapse not only the first domain, but also all of its sub-pluralities (i.e., subsets). Since the first-order domain does not contain any of these sets, this means that the axiom thus stated turns out false.

[33] The symbol '¶' was first introduced in Montague, Scott, and Tarski (unpublished) and comes back in Scott (1974).

[34] Contrary to Button's LT we don't add the qualification "if it exists" because P-COMP is enough to grant the existence of this (impredicative) plurality.

This is a basic consequence of the fact that the theory we are about to state actually is a theory of levels with one level of classes (see §5.5.), as anticipated at the end of Montague, Scott, and Tarski (unpublished, §22). That is, all the levels are bounded except the last one, which is the single unbounded level that covers the whole underlying plural hierarchy.

We are now ready to state Plural Level Theory.

Definition 4.7. PLT is the theory formulated in $\mathcal{L}_{\prec,\ltimes}$ with the following non-logical axioms: EXT_p +

$$\forall xx(\exists ss(\text{LEV}_\beta(ss) \wedge xx \preccurlyeq ss) \to \exists a(xx \ltimes a)) \qquad \text{(P-COLLAPSE)}$$

$$\forall a \exists ss(\text{LEV}(ss) \wedge a \prec ss) \qquad \text{(P-STRAT)}$$

P-STRAT is a plural analogue of the stratification principle that characterizes all axiomatizations of the conception: *every set lives at some plural level*, or *the plural level-hierarchy covers the whole universe of sets*. P-COLLAPSE is our "main engine of set production" (Yablo, 2006) and provides the final sharpening of Cantor's consistent multiplicities. The following definition parallels Def. 3.6 on the other side of Kanamori's notable inversion, offering an explicit characterization that does not hinge on ordinal primitives:

Definition 4.8 (CONSISTENT MULTIPLICITIES$^{\text{L}}$). $\mathfrak{C}^{\text{L}}(xx) \leftrightarrow_{def} (\exists ss : \text{LEV}_\beta)xx \preccurlyeq ss$

That is, *consistent multiplicities* just are *pluralities bounded by bounded-levels*. Here we can interpret the two main axioms of PLT as providing the "*if*" (P-COLLAPSE) and the "*only if*" (P-STRAT) part of the Cantorian claim that a multiplicity forms a set if and only if it is consistent. Moreover, the left-to-right direction provides a parallel improvement of the principle mentioned at the end of §3.3.: $\forall xx(\mathfrak{C}^{\text{L}}(xx) \to (\exists ss : \text{LEV}_\beta)xx \preccurlyeq ss)$, i.e., *every consistent multiplicity appears at some (bounded) plural level*, or *the hierarchy of (bounded) plural levels pins down exactly the consistent multiplicities*. In fact, PLT can do more and prove that (bounded) levels are well ordered by plural inclusion '$\preccurlyeq$'. This means that the (bounded) levels really pin down all consistent multiplicities as generating the well-founded hierarchy of sets.[35]

5. Some results in PLT

Now that we have stated the theory, let's survey some of the most interesting results to be obtained in it.

5.1. The Fundamental Theorem of PLT

Let's start from what Button labels the *fundamental theorem*, namely the result I just mentioned about the well-ordering of the levels:[36]

Theorem 5.1. *Plural levels are well ordered by* $\preccurlyeq$.

[35] To avoid specifying it every time, from here on I will talk of levels in general and take it to be clear from context, if not specified otherwise, when I mean "bounded level".

[36] I am especially grateful to Tim Button for his helpful insights on the general strategy for the proof and, in particular, for a Lemma that got me stuck for months (see fn. 66).

The reason for the label is that Th. 5.1 yields both Foundation and $\in$-induction.[37] As remarked by Wang: "The axiom of foundation sharpens the concept of iteration" (Wang, 1974, p. 216). This becomes evident especially after Scott's first proof of this theorem, a result later celebrated by both Boolos (1984, 1989) and Button. The derivation of Foundation is the real hallmark of the Iterative Conception, which provides a uniform structure within which sets are arranged. The remarkable fact about the axiomatizations of the conception is that they induce this fundamental structure by a simple and explicit description of the layers of the hierarchy. That is, instead of assuming the principle at the outset, theories of levels incorporate it in the core definition of a level. Moreover, going back to Forster's description of the conception, this result shows that the final set-up really is uniform, in that the hierarchy turns out to be well-founded even though the set-generator is changed. To use Scott's own words: "This at first surprising result shows how little choice there is in setting up the type hierarchy" (Scott, 1974, p. 210).

The details of the proof of Theorem 5.1 are a bit tedious, although they differ a bit from Button's proof for reasons of bookkeeping. I therefore defer them to Appendix 1.1.. Still, there are two interesting details about the proof. First, it can go through in a theory weaker than PLT, where P-COMP, is restricted:

Definition 5.2. PLT^{-} is PLT with P-COMP substituted by

$$\forall xx(\exists yy\forall u(u \prec yy \leftrightarrow (u \prec xx \wedge \phi(u)))) \qquad \text{(P-SEP)}$$

This is relevant because it makes the fundamental theorem available also to those scholars, like Florio and Linnebo (2021), that propose a weaker logic for plurals. Second, it is the collapse predicate '$\ltimes$' that carries over the heavy work of providing a satisfactory notion of minimality (see fn. 37), which offers an interesting insight in the kind of cross-type phenomena at the core of this approach.

5.2. Models of PLT

Let's consider the models of PLT. After Boolos (1985), these are interpreted as the usual models for full second order logic, where the power-set of the first-order domain acts as the domain for the second-order (i.e., plural) variables.[38]

Fact 5.3. $\langle V_\alpha, \mathcal{P}(V_\alpha), \ltimes\rangle \models PLT$ for all $\alpha > 0$

This means that PLT is maximally neutral with respect to the pluralities, and hence the sets, it generates. Although this may seem as a shortcoming of the theory, it is not. The reason is that this maximal level of neutrality, or, better, generality, coheres with the original spirit with which these theories of levels where originally formulated:[39]

> We could say that in the discussion above we were interested in the set-theoretical sentences true in relational systems of the form $\langle R(\alpha), \in_{R(\alpha)}\rangle$, for various ordinals α. [...] The problem we now wish to discuss is *which sentences are true in all*

[37] To be precise the result that sanctions both are two similar statements involving '$\ltimes$' and needed to derive the theorem. However, since the collapse predicate would act only as a "trans-type" kind of well-ordering, I focus on the more familiar "trans-type" notion. See Appendix 1.1..

[38] As noted by Burgess (2004, §9), there is little point in complaining against these "official" models in favor of "more natural" purely plural models, since Boolos showed that the two ways of conceiving models for plural logic are equivalent given P-COMP and the Axiom of Separation for sets. Therefore, despite the naturalness of plural models, for reasons of readability and ease of exposition, I use the official set-theoretic models with no loss for the Cantorian nature of the project.

[39] The same remarks are found in published form in Montague (1965).

> *systems $R(\alpha)$, where α is a non-zero ordinal*. The usual situation involving Gödel's Incompleteness Theorem arises here, and it can be shown that the set of sentences common to all systems $R(\alpha)$ cannot be axiomatized by a recursive set of axioms. Even so, we can present *a short and simple set of axioms adequate for the main properties of these systems*. (Montague, Scott, and Tarski, unpublished, p. 160, my emphasis)

That is, the mathematical purpose of a theory of levels is to provide a maximally general framework with respect to the set-theoretic landscape it describes. We can thus interpret PLT as a plural realization of the original project also embodied by Button's LT.

Considering models for PLT also allows us to justify the restriction on P-COLLAPSE. Consider the model of PLT for $\alpha = \omega$, which can be represented as follows:

$$V_\omega = \{\underbrace{\ldots\ldots\ldots}\}$$

$$\asymp$$

$$\mathcal{P}(V_\omega) = \{\overbrace{\ldots\ldots\ldots}\,|\,\ldots\ldots\ldots\,|V_\omega\}$$

The brackets and the pipes serve to visualize and separate the "collapsable" part of the second-order domain from its "uncollapsable" part and also from its maximal element, namely the whole first-order domain. This is a nice example to illustrate the restriction on P-COLLAPSE: if we had that any level-bounded plurality whatsoever collapses into a set, then the whole uncollapsable part of the second-order domain (and the first-order domain itself!) would collapse into sets of the first order domain, which clearly cannot be the case. Moreover, it also that there always is a last plural level (not necessarily a last bounded), namely the first-order domain itself, which PLT sees as a level or, more precisely, as its sole unbounded level.[40]

5.3. *PLT and Set Theory*

It is now time to check how PLT interacts with a theory of sets like Zermelo-Fraenkel set theory (ZF). This is the main philosophical reason behind the axiomatizations of the conception: providing a more natural and intuitive justification of the axioms of set theory. The same should be said for our Cantorian theory of levels, which can be reconnected to the developments of axiomatic set theory that followed Cantor. To justify ZF, however, we are forced to augment PLT with suitable bolt-ons to obtain those principles that the theory alone cannot sanction. For the moment, this is all the theory can do:

Proposition 5.4. $PLT \vdash$ SEPARATION, UNION, FOUNDATION

The first two additional principles are a straightforward adaptation from Button (2021), where all plural variables stand for bounded plural levels:

$$\forall ss \exists tt (ss \precnsim tt) \qquad \text{(P-END)}$$

$$\exists ss[(\exists qq \precnsim ss)(\forall qq \precnsim ss)(\exists rr(qq \precnsim rr \precnsim ss))] \qquad \text{(P-INFINITY)}$$

Namely, there is no last level and there is an infinite level, which yields the following result as for LT:

[40] In algebraic terms this is a simple consequence of the fact that, since the standard second-order domain is always the power-set of the first-order domain it always induces an algebra, namely the power-set algebra, whose top element is the first-order domain itself.

Proposition 5.5.

(i) PLT + P-END $\vdash$ PAIRING, POWER-SET
(ii) PLT + P-END + P-INFINITY $\vdash$ INFINITY$_Z$[41]
(iii) PLT + P-END + $\neg$P-INFINITY $\vdash$ ZF_{FIN}[42]

The second principle is a bit more complex as it derives Replacement, a notoriously troublesome principle in the context of the Iterative Conception.[43] In LT this is captured through full second-order means. That is, for any functional F:

$$\forall F \forall a (\exists s : \text{LEV})(\forall x \in a) Fx \subseteq s \qquad \text{(UNBOUNDED)}$$

Its intended meaning is that the the hierarchy of levels is so tall that no set can be mapped unboundedly into it. Despite the fact that we have full P-COMP and so the plural reading of full second-order logic from Boolos (1984, 1985), we cannot go for a straightforward translation of Button's principle. Simply translating 'Fx' as '$x \prec xx$' would not be enough because we would still be lacking a mapping from a to xx. That is, it is true that P-COMP allows us to define arbitrarily big pluralities that may even exceed the hierarchy of bounded plural levels. However, '$x \prec xx$' does not capture the same meaning of 'Fx' even though they are both generated by an unrestricted comprehension principle. The reason is that, being functional, that the latter already embodies the concept of a mapping.

A first reply to this issue would be bite the bullet and go schematic and re-state Button's principle for each functional formula 'ϕ'. However, I do not think that this is satisfactory, precisely because of Boolos' reading of plural-logic, which places it on a par with second-order quantification while avoiding its shortcomings. In other words, since the logic of plurals does not lack expressive resources, there must be a way to make sense of quantification over functions in this context. This solution, it turns out, is simply to properly exploit the power of P-COMP to single out the notions of a functional plurality 'ff^a' over a set 'a' and its plural image '$ff^{[a]}$'. This is allowed simply by the above result concerning the derivation of the fragment of ZF and by the comprehension principle. However, since the details are more a matter of bookkeeping rather than being really interesting, I leave them to Appendix 1.2.. In the end, the principle looks like this:

$$\forall a (\forall xx \approx ff^a)(\exists ss : \text{LEV}_\beta) ff^{[a]} \preccurlyeq ss \qquad \text{(P-UNBOUNDED)}$$

In other words, the plural image of any functional plurality over a set is consistent and can thus form a set.[44]

The final result is again straightforward after LT's analogous theorem:

Theorem 5.6. *Let* PLT^+ *be* PLT + P-INFINITY + P-UNBOUNDED. *Then* $PLT^+ \vdash ZF$

[41] There is a set containing the empty set and closed under successor.
[42] See Button (2021, fn. 27).
[43] See Boolos (1971, 1989); Potter (1990, 2004).
[44] Moreover, note that an alternative route to obtain both Infinity and Replacement is the one through Reflection Principles first pursued by Scott (1974) in the context of the Iterative Conception. However, since this approach has already been pursued by Burgess (2004), who interprets it as way to make sense of a limitation of size view, I defer the discussion of this principles to the next section.

5.4. PLT vs. LT

The reason why the above results are straightforward is due to the relation between PLT and its set-theoretic second-order cousin LT:[45] the two theories are synonymous in the sense of being definitionally equivalent. Once again thanks to Boolos (1984, 1985), this result is straightforward: one just interprets the second-order variables of LT along the plural interpretation (see next section for the primitive predicate '$\ltimes$').[46] Although this equivalence may be interpreted as a trivialization of PLT as a mere plural copy of LT, I think this thought is misguided.

For one thing, PLT originates from a completely different standpoint which makes my project and Button's quite different. On the one hand, I want to provide a theory of Cantorian "consistency" (or co-existence, see §6.2.) for pluralities that sharpens Cantor's idea that sets are obtained by collapsing pluralities. To do so, I employ the tools provided by the Iterative Conception since this route to consistent multiplicities had not been explored yet. On the other hand, Button moves already within the framework of the conception and his appeal to full second-order logic is not substantial, but only instrumental to obtain the quasi-categoricity theorem. That is, LT does not put the same weight as PLT on higher-order variables simply because it already is a theory of sets that could work also if formulated in first-order terms. On the contrary, PLT is grounded on a substantial appeal to pluralities to generate sets along an iterative process which is instrumental to grant the consistency of the plural-to-set abstraction.[47] On top of this, I think that this is a rather comforting result. The reason is that, no matter how one finds the plural-talk or the collapse idea extravagant and exotic, pushing back against PLT becomes quite hard if one is also provided with a "safe" and "familiar" retreat into a standard second-order theory of sets like LT.

That said, since we mentioned that Button's second-order formulation of LT is instrumental for obtaining quasi-categoricity, it should not be surprising that the same result carries over also to PLT. This is another crucial result in these theories, along with the well-ordering of the levels, first proven by Montague, Scott, and Tarski (unpublished) and later perfected in Button and Walsh (2018), who also add the qualification of internal categoricity.[48]

5.5. Classes in PLT

One of the chief advantages of retrieving Cantor's picture through PLT is that, together with a conception of sets, it also legitimizes the approach to proper classes presented in the seminal Uzquiano (2003). This has two positive consequences. First, Uzquiano's main point is that plural logic make sense of class talk, especially in its impredicative form as axiomatized in Morse-Kelley class theory (MK).[49] This is not only useful, but sometimes indispensable to formulate

[45] LT is natively a second order theory so we do not need to flag it. It's first order counterpart is tagged as LT_1 instead.

[46] Alternatively, we could plug in a collapsing predicate for LT's second order variables, modify it along the same lines of PLT and the result would be even more perspicuous.

[47] This is also why the suggestion from fn. 46 of equipping LT with a collapsing predicate to make the equivalence with PLT more perspicuous is not so trivial. It would rather carry over a substantial commitment to a certain view regarding the process of set formation.

[48] Internal categoricity being the object-language claim, derivable from second-order logic, that there is a relation that can internally code (or mimic) the behavior of an isomorphism between two models. See Button (2021, §6) who also strengthens to full internal categoricity. I redirect the reader to the aforementioned texts for the proofs since here there are no relevant differences between the two cases.

[49] Here we shall be careful to interpret MK without a principle of limitation of size *à la* von Neumann (1925, 1928), i.e., *all proper classes have the same size as the universe*. The reason is that PLT is not equipped with a choice-like axiom in a plural guise after Burgess (2004). Here I agree with Boolos' on AC not being obviously entailed by the Iterative Conception, while disagreeing on the same view concerning Replacement and being closer to Shoenfield (1967, 1977). However, caution should be used even if we accept Burgess' Plural Choice since this must be equivalent to Global Choice for the theory to be equivalent to MK plus limitation of size (see Fraenkel, Bar-Hillel, and Lévy, 1973). This is not immediately obvious since Burgess claims that his axiom is equivalent to

certain set-theoretic statements like large cardinal hypotheses.[50] Second, this completes the interpretation of Cantor's passages, where both consistent and inconsistent multiplicities are part of the picture. In PLT level-bound pluralities collapse into sets and unbounded pluralities do not. In class terms these correspond, respectively, to sets (*level-bound pluralities*) and proper classes (*level-unbounded pluralities*), which naturally matches Cantor's distinction between consistent and inconsistent multiplicities (see §6.1.).

Furthermore, PLT provides a natural interpretation of Uzquiano's notion of "correspondence" between a class and a set (p. 74), namely our primitive predicate '$\ltimes$'. More specifically, had we taken the usual set-membership predicate '$\in$' as primitive rather than '$\ltimes$', we could have defined the latter in the same way Uzquiano defines the correspondence between a set and a class: $xx \ltimes x \leftrightarrow_{def} \forall y(y \in x \leftrightarrow y \prec xx)$.[51] In particular, the current approach is enough to provide the plural reading of two-sorted MK mentioned by Uzquiano.[52] This further yields a perspicuous reading of Separation and Replacement as full axioms rather than schemas and further clarifies the seemingly weird notion of "functional plurality" invoked above simply as a functional class.

Moreover, this has the remarkable consequence of establishing PLT as a realization of the rank-free theory with classes originally proposed by Montague, Scott, and Tarski (unpublished), which is the first complete instance of a theory of levels. That is, after proposing their theory for sets (of which Button's LT is the latest and most developed descendent), Montague, Scott and Tarski speculate that one could provide a similar theory of levels with n levels of classes, where $n = 1$ would be equivalent to MK.[53] However, as for sets, their realization is rather convoluted and not so easy to follow. Therefore, in light of the above remarks concerning the reading of plurals as classes and PLT's unique unbounded level, i.e., class-level, we can say that our theory realizes the aforementioned project in a more perspicuous way, just as Button's LT is a more straightforward realization of their original theory of levels.

6. Other Approaches

PLT is the alone in trying to make sense of Cantor's remarks concerning pluralities and sets. In this section we examine rival approaches and argue for PLT as making better sense of set formation as a process of plural-to-set abstraction.

6.1. Limitation of Size

As mentioned in the opening, the project of making sense of Cantor's remarks through to the Iterative Conception of Set is compatible with Cantor's original ideas although it is probably

standard set-theoretic AC. However, since the proof happens against the background of a theory which includes a plural (read "second-order") reflection principles, there is still room to believe the axiom to be a version of global choice. This is enforced by the plural reading of classes and the fact that Burgess' axiom states the existence of a choice function over pluralities *simpliciter*, where these are generated by P-COMP. This makes the case for plural choice as a logical principle a bit troublesome, unless one subscribes to a limitation of size view of the process of set formation as Burgess does (see §6.1.).

[50] Although other strategies may be available, such as going property-theoretic or taking class-talk at face value, the plural strategy seems the most promising and less problematic, especially after Boolos' work. Thanks to an anonymous referee for pointing out these alternatives to me.

[51] This is also the way in which LT would interpret PLT.

[52] See Uzquiano (2003, p. 78). While the plural interpretation of two-sorted MK is immediate (set variables are mapped to sets and class variables are mapped to pluralities), one-sorted MK has to be translated to the former for the result to go through. While the transfer is almost trivial, as noted by Uzquiano himself, given the categoricity of PLT, the correspondence with one-sorted MK may not be so trivial. I leave this to further work.

[53] This would seem to go against the usual view, also endorsed by Uzquiano (2003), that classes do not iterate. However, as the authors remark: "Further, the theories obtained by setting n [i..e, the number of iteration of classes] equal to 2 or 3, though unknown in the literature, can be quite useful for the development of certain branches of mathematics, for instance, parts of algebra and model theory". (Montague, Scott, and Tarski, unpublished, p. 178). It is an interesting matter for future work to understand what commitments (ideological or ontological) this further levels of classes force us into, especially in light of the plural reading of classes and of the recent debate on super-pluralities and its possible elimination through plural cover predicates (see fn. 32).

not faithful to its original intents (Ferreirós, 2007). As thoroughly documented by Hallett (1984) it makes more sense to attribute him a *Limitation of Size* Conception, traditionally opposed to the Iterative Conception.[54] A nice way of characterizing the two approaches comes from Uzquiano (2003) by adapting his talk of "from above" and "from below" principles (pp. 70-71). Starting from the Iterative Conception, the whole idea is to axiomatize a constructional procedure that builds sets *from below* by iterating the application of some operation. Then, one may ask if there are collections so big that they escape this process of stratification. In a sense, these would be "too big from below" and PLT characterizes them as the opposite of "consistent multiplicity":

Definition 6.1 (INCONSISTENT MULT.). $\neg\mathfrak{C}^{\mathrm{L}}(xx) \leftrightarrow \neg(\exists ss : \mathrm{LEV}_{\beta})xx \preccurlyeq ss$

That is, PLT describes inconsistent multiplicities as those pluralities that are level *unbounded*, i.e., that cannot be built "from below" by iteration or that are "un-stratifiable" so to speak: *if, running through the levels (from below), none of them binds xx, then xx is inconsistent*, i.e., it is a proper class.

On the other hand, Limitation of Size determines *from above* what is a set or not, depending on whether a collection can or cannot be put into one-one correspondence with collections, like the universe of sets (von Neumann, see fn. 49) or the ordinals (Cantor, see Hallett, 1984). Here the idea is the opposite, namely "too big from above". These collections are not too big because they escape a process of stratification that starts from below, but are rather given in advance, i.e., from above, as too big and then are used to characterize sets as small collections (so perhaps here the correct label is "small from above"). In a sense, the two conceptions reverse each other's order of explanation, focusing either on sets or classes first: the Iterative Conception builds sets from below and then says that classes escape the stratified structure of the set-universe; Limitation of Size first assumes classes as big collections and then uses them to pin down sets as small collections. In other words, the contrast is a matter of *conceptual priority between sets and classes*.

According to Burgess (2004), the idea of limitation of size is also captured by reflection principles, originally described as from above limitations by Uzquiano: properties of the entire universe are reflected down to initial segments of the hierarchy of sets. It is on these principles that Burgess builds his theory BB (Boolos-Bernays) as a plural theory of sets that should capture the Limitation of Size Conception.[55] In particular, the plural reading of classes is essential for Burgess as it enables a principle of class-reflection *à la* Bernays (hence the name) suitable to derive not only the axioms of Infinity and Replacement, but even large cardinal principles up to Mahlo. The principle so interpreted also makes sense of the Cantorian distinction by defining *inconsistent multiplicities* as those pluralities such that "... any statement Φ that holds of them continues to hold if reinterpreted to be not about all of them but just about some of them, few enough to form a set" (Burgess, 2004, p. 205).

So interpreted, I do not think that Burgess' characterization is a satisfactory account of the Cantorian view. The reason is that, it already presupposes the notion of set and so it does not go much further than Cantor in simply stating that consistent multiplicities are those pluralities that do (and inconsistent those that do not) form sets. On the contrary, PLT explains the distinction on the basis of a proper description of the process of set formation as arising from plural abstraction. That is, consistent multiplicities are not just pluralities that form sets, but rather pluralities that are bounded by the (explicitly defined) levels of the cumulative

54 see Wang (1974, ch. 6) and Hallett's critique of the Iterative Conception in his §6.1.

55 See also Pollard (1996) on this same link between pluralities and limitation of size.

hierarchy, while inconsistent multiplicities are those that escape these boundaries. To put it in more perspicuous terms: *PLT first explains the notion of (in)consistency as level (un)boundedness and then says that consistent multiplicities are those that form set, it does not just say that some multiplicities are consistent because they are those that form sets*. While I do not take this to be a definitive argument in favor of the Iterative Conception over Limitation of Size, I nonetheless interpret it as explicitly favoring the former to make sense of Cantor's ideas concerning the formation of sets and the contrast with classes.

On top of this, there are two further considerations. First, despite clearly being "from above", reflection principles are not an exclusive of the Limitation of Size Conception. For instance, Scott (1974) uses them to derive Infinity and Replacement in his axiomatization of the Iterative Conception.[56] Although it is true that his reflection is first-order while Burgess' crucially is second-order (i.e., plural), it is still possible to envisage a plural-reflection principle *à la* Scott based on the relativization of a formula to a plural-level. This would seemingly preserve a certain amount of the "from below" spirit although the case for reflection principles within the Iterative Conception, in my opinion, remains controversial and in need of further clarification (see fn. 56). Moreover, the adoption of these principles should still be taken with caution. On one hand, Linnebo (2007) shows how, from assumptions that Burgess accepts in his use of reflection, one can easily derive that every plurality forms a set, which is inconsistent with the core assumption of P-COMP, as we explained in the opening. On the other hand, third-order reflection has been proven to be inconsistent[57] and therefore, assuming he could overcome Linnebo's challenge, Burgess may still fall prey of something like the Bad Company Problem for Neo-Fregeans.

Second, it is precisely a response to the Bad Company Problem that could reconcile the limitation of size component of Cantor's view with an iterative picture. Ironically, this theory should be labelled BBB (Boolos-Boolos-Boolos) as it is based on pivotal contributions to three areas: plural logic, the Iterative Conception and Neo-Fregeanism. In fact, Boolos (1987, 1989) retrieves Frege's program through a consistent reshaping of the infamous Basic Law V, called "New V", grounded on the notion of a *small concept*. This could capture Cantor's idea of consistent multiplicities, once the plural reading of second-order logic from Boolos (1984, 1985) has been implemented. Then, he contrasts this approach to the Iterative Conception (Boolos, 1971, 1989) in the usual terms of *logical* versus *combinatorial* collections (see §3.2). The former notion, he argues, has more traction to derive Separation, Replacement and Choice in their schematic or conceptual formulation. Since this fragment of ZFC is precisely what poses problems for the iterative-combinatorial picture, one may want to reconcile the two ideas after the pioneering work of Shapiro and Weir (1999). The two conceptions could work together to

[56] Scott was explicitly inspired by Lévy's seminal work (1960; 1961), while traces of the link between the conception and reflection can already be found in some remarks from Gödel (see Wang, 1977, p. 325, Wang, 1996, p. 285 and Koellner, 2009). However, one may also argue that Scott's appeal is illegitimate due to reflection principles clearly matching Uzquiano's "from above" characterization. Here I leave the question open, although I think that figuring out the status of reflection and related principles within the Iterative Conception is one of the most pressing issues in the philosophy of set theory, especially after the most recent developments of large cardinals and inner model programs. These seems to suggest an intrinsic link between the logical and the combinatorial component of the set concept, the former being more clearly represented by reflection. See in particular Bagaria and Ternullo (2025); Ternullo and Venturi (MS) who comment on this topic in connection to the most recent results in the search for new axioms (Aguilera et al., 2025; Aguilera, Bagaria, and Lücke, 2024).

[57] See Koellner (2003, 2009); Tait (1998, 2005).

smoothly retrieve the whole of ZFC from a fully Cantorian (plural-based) conception of set that keeps track of both the *logical-limitative* and of the *combinatorial-iterative* components.[58]

6.2. Potentialism and Critical Plural Logic

A second approach that must be mentioned is the one that pursues a modal understanding of Cantor's remarks, as in Øystein Linnebo's project of set-theoretic potentialism (2010; 2013).[59] Going back to Cantor's initial quotes, a perspicuous way of making sense of them is by the slogan "*set-existence* is a matter of *co-existence*" (Roberts, MS). However, this in not enough and the notion of co-existence needs a further sharpening: after all, under the common reading of plural logic, all the sets can co-exist as a plurality, but we don't want them to collapse into a set. In the recent debate a popular sharpening has been provided through a modal analysis that takes Cantor's use of modal expression at face value and re-interprets the slogan as: "*possible*" set-existence is a matter of "*possible*" co-existence.

The purported advantage of this approach, according to Roberts (MS), is that it does not modify the notion of co-existence, but rather takes the notion as it is and analyzes it in a modal context to make sense of it. This analysis ultimately yields a restriction of P-COMP, since its modal translation

$$\Diamond\exists xx\Box\forall x(x \prec xx \leftrightarrow \phi(x)) \qquad (\text{P-COMP}^{\Diamond})$$

is not true for all conditions 'ϕ' but only for those that are "*extensionally definite*" (Linnebo, 2010, p. 157). Therefore, contemporary versions of modal set theories Sutto (2024), disqualify a priori pluralities such as those of all sets, ordinals or cardinals. This is in stark contrast with PLT, whose endorsement of P-COMP makes it distinctively actualist. In this sense, PLT represents the opposite view on co-existence as framed by Roberts' slogan: *set-existence* is a matter of *co-existence* at a stage or, better, *at a plural level*.[60]

If one finds the appeal to modalities problematic,[61] Critical Plural Logic (CPL) as advanced by Florio and Linnebo (2021) provides a non-modal theory grounded on the same idea of limiting P-COMP to make all multiplicities consistent so to speak. More precisely, Florio and Linnebo black-box the notion of "extensional definiteness" (i.e., consistency in Cantor's informal terms), which can instead be explicitly defined by potentialism, and axiomatize it. Their approach interestingly resembles Zermelo's original axiomatization of set theory if interpreted as black-boxing a consistent notion of collection and axiomatizing it.

No matter how one frames it, an issue with this approach is that it seems to be in stark contrast with Cantor's opening passages where he does not deny that inconsistent multiplicities are somehow conceivable or that they exist. What is outside the scope of "mathematical contemplation", in the passage from the letter to Hilbert, are the sets obtained from those pluralities, which are straight-away contradictory. The multiplicities themselves, on the contrary, are explicitly said to be "*definite*" (Cantor, 1899), a position that seems to put Cantor in agreement with unrestricted P-COMP. But what about co-existence then? If one forces

58 This would also agree with the direction that contemporary set theory seems to be taking in some of its latest developments mentioned at the end of fn. 56. However, it would also pose a question on the status of pluralities *qua* combinatorial or logical collections, since this operation seems to require them to instantiate both features. The situation is similar to the one outlined by Maddy (1988) concerning her previous 1983 work on classes.

59 The other main proponent of potentialism is Studd (2013, 2019). However, since he does not place Cantor and plural logic at the center of his investigation, here I focus on Linnebo's work.

60 Under this respect I do not agree with Roberts (MS) in opposing the Iterative Conception to both potentialism and the limitation of size view, but I rather interpret potentialism as a modal approach to the conception in accordance with Sutto (2024).

61 See Sutto (2024) for a survey on issues concerning modal approaches to potentialism and Button (MS) for a very careful critique.

the debate in terms of this notion, which may not be accepted by those who endorse a limitation of size view,[62] there seems to be no better way than framing the idea as *co-existence at a plural level* as articulated by the Iterative Conception *qua* instantiated by PLT.

However, despite Yablo's (2006) observation that favoring P-COMP is the route generally taken when the conflict with COLLAPSE is outlined, a fully articulated theory of sets in agreement with the former principle had yet to be explored. The reason is that most axiomatizations of the Iterative Conception are first-order and completely overlook the Cantorian conception of the process of set construction as plural-to-set abstraction. On top of that, Florio and Linnebo seem to further deny that such an account is philosophically defensible:

> We have described two very attractive applications of plural logic: as a way of giving an account of sets, and as a way of obtaining proper classes "for free". Regrettably, it looks like the two applications are incompatible. [...] Is there any way to retain both of the attractive applications of plural logic? To do so, we would have to restrict the domain of application of the "set of " operation so that the operation is undefined on the very large pluralities that correspond to proper classes, while it remains defined on smaller pluralities. The obvious concern is that this restriction would be *ad hoc*. (Florio and Linnebo, 2021, p. 72)

Against this, PLT shows that the two applications are compatible after all. Moreover, it does so in a non *ad hoc* way as the restriction on '$\ltimes$' is motivated by the Iterative Conception of Set, an intuitive and natural way of describing the process of set formation. Therefore, unless one wants do deny the naturalness of the conception, PLT seems to constitute a natural reply to the challenge posed by Florio and Linnebo. Remarkably, arguing against the conception is not an option for them, since they too seek inspiration in the famous Gödel's passage with which we started:

> To respond to this challenge, we might seek inspiration from Gödel, who points to a restriction when he requires that the "set of " operation be applied to "well-defined objects". [...] One option is to understand Gödel as requiring that the objects in question be properly circumscribed. [...] there are indeed "collections" that fail to be properly circumscribed. However, we also argue that every plurality is (in the appropriate sense) properly circumscribed and can thus figure as an argument of the "set of"operation. (Florio and Linnebo, 2021, p. 72)

As shown by PLT, a natural way to make sense of Gödel's claim is by interpreting "properly circumscribed" as "level-bound", in agreement with the developments of the Iterative Conception that stems precisely by that passage. Making all pluralities properly circumscribed seems to be an additional, rather substantial, assumption tied to an implicit potentialist understanding of the process of set formation. While challenging this approach is beyond the scope of this paper, I think that the contrast between PLT and CPL highlights an *interesting tension between two ways of interpreting the Cantorian plural conception of set and that ultimately boils down to the conflict between actualism and potentialism*. On my end, I argued that PLT does justice to an approach that seems to be more in line with Cantor's remarks, but that has not been hugely

[62] Co-existence as smallness does not seem a perspicuous explanation.

debated for various reasons, such as the focus on first-order axiomatizations of the Iterative Conception, the development of potentialism itself and the connection between plural-based axiomatization of sets and the Limitation of Size Conception as proposed by Burgess (2004); Pollard (1996). Of course this is far from settling the dispute, but at least the kind of actualism more in line with the Iterative Conception rather than the Limitation of Size view has now been provided with a carefully articulated theory that speaks the same plural-based language of potentialism.

6.3. Cantorian Set Theory

Another reason why an approach like PLT struggled to emerge, I think, is due to the only development of a plural iterative conception before mine being the one advanced by Oliver and Smiley (2016, 2018). More precisely, the fact that they explicitly place their Cantorian theory in opposition to standard axiomatic approaches like ZF and ZFC obscured the fact that such an account could also serve the purposes highlighted in this paper: making sense of Cantor plural remarks while being faithful to a conception of set tied to standard axiomatizations.[63] Moreover, since PLT could in principle be interpreted as a development of their own view, I argue that it also provides substantial improvements that go beyond the fact that it can derive pure axiomatic set theory. That is, PLT would perform better even if it was made to serve Oliver and Smiley's critique of the empty and singleton sets.[64] In particular, I think that they missed on three occasions to make their plural account of the conception "truly plural" so to speak.

First, despite the fact that they open the chapter on the theory by observing that "a great deal of reference to sets is merely an unnecessary and obfuscatory way of speaking [...] fuelled by the singularist drive to replace plural language by talk about sets" (2016, p. 245), their version of '¶' outputs sets rather than pluralities. That is, while their histories are pluralities, their potentiation applies to a plurality and produces a set. Therefore, their levels (still potentiations of histories) end up being sets rather than pluralities. I think that this is a substantial missed opportunity on having a completely plural characterization of the hierarchy of sets as PLT has. For instance, as noted above, the fact that our levels are pluralities allows to interpret PLT as a realization of a theory of levels with 1 level of classes which recaptures MK as speculated by Montague, Scott, and Tarski (unpublished). This is obviously something that is out of reach for a theory with only set-levels.

Second, rather than a primitive relation like my (and Burgess') '$\ltimes$', they start with the functional term '$\{\}$' which is the same as my '$\uparrow$'. While this also captures Gödel's "set of" operation, their endorsement of full P-COMP makes it denote a partial function, but they offer no explanation of when it may be total and are instead forced to appeal to a free logic with existence predicates both for plurals and for singular entities. That is, they do not realize they can use the level-theoretic setting to make sense of Cantor's partition between consistent and inconsistent multiplicities and also preserve classicality. To do so, however, appealing to a relational symbol is quite crucial, since '$\ltimes$' does not prejudge, as '$\uparrow$' does, whether there is such a thing as the collapsed set. This, I argue, is another missed opportunity: since they care so much for being faithful to Cantor why missing the chance of making sense of one of its most relevant yet obscure distinctions?

63 This despite the fact that they too argued for a use of their theory to sanction ZF as I did. See (Oliver and Smiley, 2016, §14.8).

64 This on top of the fact that my theory is based on an updated version of the theory they use as a blueprint, namely Potter (2004). Of course it would not be fair to accuse them of missing out a theory that was not even there when they developed their own. Nonetheless, the same improvements that LT brought to Potter's theory can also be appreciated for PLT with respect to their theory.

Third, their axioms seem to also miss some relevant points. For one thing, they postulate that a set is not included among the plurality which generates it, a fact easily derived from rules on how to alternate between '$\uparrow$' and '$\downarrow$' and the pivotal proof of well-ordering. Since they also aim for this proof, and given its importance in the context of the conception, this gap in their axiomatization is quite significant. Furthermore they assume as an axiom that sets uniquely determine their elements: $\uparrow xx = \uparrow yy \rightarrow xx \approx yy$. This is the opposite of Extensionality which, they claim, is implicit in the syntactic characterization of '$\{\}$' as a function. While this is true, it also completely reverses the order of explanation, which should go from the elements of a set to the set, not the other way around, betraying the Cantorian spirit of the project. Rather than have it implicit, making the other direction explicit and assuming as an axiom the biconditional, namely the Plural Law V from Florio and Linnebo (2021), would have eased the understanding of the process of set formation.

Therefore, despite the two projects come from different backgrounds, I think there are reason to favor PLT even to frame Oliver and Smiley's Cantorian set theory.

7. Conclusion

When introducing the Iterative Conception of Set, Dana Scott starts from a question concerning the Axiom of Separation: "where does the a [to which we apply Separation] come from?" (Scott, 1974, p. 208). His answer is of course the iterative process of set-formation based on a prototypical theory of stages or levels. An alternative reply, based on a Cantorian conception of set would be: Scott's a is collapsed from a given plurality aa. However, the question immediately resurfaces: *where does the plurality aa come from*? In this paper I proposed a reply that lines with Scott's: I described an iterative process of plural-to-set abstraction where pluralities collapse into sets as they appear at some level of a plural cumulative hierarchy of sets. To do so I resorted to the axiomatization of the conception in terms of a theory of plural levels, where the notion of a level is defined in explicit and non-recursive terms. The resulting Plural Level Theory yields a perspicuous explanation of Cantor's idea concerning the process of set-formation: *consistent* and *inconsistent multiplicities*, respectively, are explained as *level-bounded* and *level-unbounded pluralities*.

Besides reconciling the idea that sets are collapsed from a given plurality with the most popular conception of set, PLT also exhibits some nice results in line with the literature that inspires it. The most important are the well-ordering of the levels and the derivation of the standard axioms of Zermelo-Fraenkel set theory, which mean that PLT actually pins down the "plural skeleton" of the Cumulative Hierarchy of sets. Moreover, if one finds the language of plurals too exotic, PLT can be traced down to an equivalent nice theory of sets, Button's LT, which is definitionally equivalent to PLT modulo Boolos' plural reading of second-order quantification. The same reading enables the plural understanding of classes advocated by Uzquiano, which means that PLT can be made equivalent to (two-sorted) Morse-Kelley class theory. This is a crucial point because it permits an account of both notions of Cantorian multiplicities in terms of pluralities while maintaining, at the same time, a fundamental and non ad hoc link between some pluralities and the sets.

Finally, I argued that PLT performs better than other approaches when it comes to regimenting the idea that sets are obtained from a process of plural-abstraction. First, although some may argue that Cantor's original view squares better with a limitation of size approach, I argued that PLT offers a more satisfactory account of Cantor's notion of multiplicity than

Burgess' BB. The best chance to make a limitation of size approach work, I also argued, is to put it side by side with an iterative view to take care of both the combinatorial and the logical aspect of the process of set formation. Second, I showed how PLT meets the challenge advanced by potentialism, which argues that a plural account of sets grounded on full comprehension is not philosophically justifiable. On the contrary, PLT not only shows that this is possible, but does so by departing from the same observation on the Iterative Conception that the non-modal account of potentialism favors. Third, and finally, I showed how PLT performs better than the analogous project developed by Oliver and Smiley. Notably this is true even if we set aside the dispute over the status of singletons and the empty set and focus solely on the core idea of pluralities as the grounds for the process of set formation.

Of course many more questions concerning PLT and its rivals remain unanswered. In particular, the conflict between actualism and potentialism, on one side, and the one between the Iterative Conception and Limitation of Size, on the other, represent pivotal crossroads in the philosophy of set theory. While it is beyond the scope of this paper to settle them, here I sketched some of the possible replies provided by PLT. My overall aim was to present and do justice to a plural-based account of the process of set-formation that, setting aside Oliver and Smiley non-standard set theory, was still missing from a literature that mostly focused on Limitation of Size (between the late 1990s and the early 2000s) and on potentialism (in the past fifteen years). Now that a plural iterative conception of set has been carefully outlined, the stage is set for a fair debate on which approach best captures the intuitions of the father of set theory.

A. Appendix

1.1. The Proof of Well-ordering

First of all, remember that the proof goes through simply by appeal to Plural Separation rather than full Comprehension. Moreover, to avoid specifying it every time, we assume that all the lemmas where we apply P-COLLAPSE below are conditionals on the levels being bounded. This can be done without loss of generality since, in general, all levels except the last one, namely the first class-level, are always bounded. In other terms, this is like saying that the well-ordering happens within the last level, which is not different from a general statement of well-ordering for a class.

Let's start with some basic facts and definitions. First, (4) and (5) from Def. 2.3 yield plural *transitivity* and *super-transitivity*, granted an analogue of (5) for sets after Button (2024): '$a \rhd b \leftrightarrow_{def} (\exists c \in b) a \subseteq c$':

Definition A.1. A set a is TRANSITIVE iff $(\forall x \in a) x \subseteq a$. A plurality aa is TRANSITIVE iff $(\forall x \prec aa) x \sqsubseteq aa$.

Definition A.2. A set a is SUPER-TRANSITIVE iff $(\forall x \rhd a) x \in a$.[65] A plurality aa is SUPER-TRANSITIVE iff $(\forall x \blacktriangleright aa) x \prec aa$.

A significant fact is then that transitivity and super-transitivity carry over from the pluralities to the respective collapsed sets:

Fact A.3. Any set collapsed from a (super)-transitive plurality is (super)-transitive.

[65] Button (2021, fn. 10) notes that the property of super-transitivity (for sets) has many different names in the literature. He chooses the label "potent" to highlight the connection with potentiation. Here I prefer to stick to "super-transitive", also adopted by Linnebo (2007), to better highlight the connection with the more familiar notion of transitivity.

Proof. Consider an arbitrary transitive and super-transitive aa and assume that $\uparrow aa$ exists. TRANSITIVITY: consider an arbitrary $x \in\uparrow aa$ and an arbitrary $y \in x$; since $x \in\uparrow aa$, $x \prec aa$ (Def. 2.3.1); therefore $y \in x \prec aa$, but aa is transitive, so $y \prec aa$ and, again by Def. 2.3.1, $y \in\uparrow aa$. SUPER-TRANSITIVITY: consider an arbitrary x and let c be a set such that $x \subseteq c \wedge c \in\uparrow aa$; by Def. 2.3.1 $c \prec aa$; therefore, $x \subseteq c \prec aa$, but aa is super-transitive so $x \prec aa$ and, by Def. 2.3.1, $x \in a$. □

The following fact about plural potentiation is trivial but worth mentioning:

Fact A.4. $uu \preccurlyeq \P\P(uu)$.

Proof. Consider an arbitrary $x \prec uu$. To have $x \prec \P\P(uu)$ means that there is a c such that $x \subseteq c \prec uu$. The fact trivially follows by instantiating c with x. □

We can then trace a connection between the notion of plural super-transitivity and of plural potentiation:

Lemma A.5. *$\P\P(aa)$ is super-transitive.*

Proof. Assume that $\P\P(aa)$ exists and consider an x such that $x \blacktriangleright \P\P(aa)$. So there is a $c \prec \P\P(aa)$ such that $x \subseteq c$, that is, $c \blacktriangleright aa$. But again, so there is a $b \prec aa$ such that $c \subseteq b$. Since $x \subseteq c \subseteq b$, we can conclude $x \prec \P\P(aa)$. □

Lemma A.6. *aa is super-transitive iff $aa \approx \P\P(aa)$.*

Proof. $\Leftarrow$ follows from Lemma A.5. $\Rightarrow$: assume aa is super transitive and $\P\P(aa)$ exists. Therefore, $x \blacktriangleright aa$ and so $x \prec aa$ by super-transitivity and Fact A.4. □

Lemma A.7. *Every level is transitive and super-transitive.*

Proof. Fix a level $ss \approx \P\P(uu)$, for some history uu. TRANSITIVITY: consider $a \prec ss$ and $x \in a$. Since $a \prec ss \approx \P\P(uu)$, $a \subseteq c \prec uu$ for some c. Since $x \in a$ and $a \subseteq c$, then $x \in c$. From Def. 4.4 $c = \uparrow(\P\P(c \cap uu))$, therefore, $x \prec \P\P(c \cap uu)$, but $\P\P(c \cap uu) \preccurlyeq \P\P(uu)$, so $x \prec \P\P(uu) \approx ss$. SUPER-TRANSITIVITY: use Lemma A.6. □

Up to now everything matches the proof in Button (2021). Before proving minimality, as LT does, some results take care of the type differences through '$\ltimes$'.[66]

Lemma A.8. *If every Φ is super-transitive, some uu are Φ and there is a level ss such that $uu \not\preccurlyeq ss$, then there is some tt which is a $\ltimes$-minimal Φ: $\Phi(tt) \wedge \forall aa(\Phi(aa) \rightarrow \uparrow aa \not\prec tt)$.*

[66] Special thanks to Tim Button for pointing me out this crucial passage, without which the proof remained stuck for months.

Proof. Assume some uu such that $\Phi(uu)$ and assume some level ss such that $uu \precnsim ss$. Apply P-SEP twice:

$$cc :\approx \|x \prec uu : \forall ss((\Phi(ss) \land \textsc{Lev}(ss)) \to x \prec ss)\| \approx$$
$$\approx \|x : \forall ss((\Phi(ss) \land \textsc{Lev}(ss)) \to x \prec ss)\|$$
$$dd :\approx \|x \prec cc : x \notin x\|$$

Note that $dd \preccurlyeq cc \preccurlyeq uu \precnsim ss$ and since $\textsc{Lev}(ss)$ we can apply P-COLLAPSE to obtain $d =\uparrow dd$. Of course $d \not\prec cc$, otherwise $d \in d \leftrightarrow d \prec dd \leftrightarrow d \notin d$, which is absurd. So there must be some tt such that $\Phi(tt)$ and $d \not\prec tt$. For reductio assume there is a vv such that $\Phi(vv)$ and $\uparrow vv \prec tt$. Since $d \subseteq\uparrow vv$ and since every Φ is super-transitive we have $d \prec tt$, contradiction.

□

Lemma A.9. *If hh is a plural history such that $hh \preccurlyeq ss$ for some level ss and $a \prec hh$, then $\downarrow a$ is a level.*

Proof. For reductio assume there is a history hh and a level ss such that $hh \preccurlyeq ss$, but that the conclusion of the lemma does not follow. Before moving on note the following corollary:

Corollary A.10. *If $a \prec hh$, then $\downarrow a$ exists.*

Proof. Assume $a \prec hh$. Since $hh \preccurlyeq ss$, $a \prec ss$ and since every level is transitive $a \sqsubseteq ss$. Apply P-SEP: $aa \approx \|x : x \prec ss \land x \in a\| \approx \|x : x \in a\| \approx\downarrow a$. □

Therefore we can now apply Lemma A.8: fix some $a \prec hh$ such that $\downarrow a$ is a $\ltimes$-minimal non-level, i.e., $(b \prec hh \land \textsc{Lev}(\downarrow b)) \to b \not\prec\downarrow a$. This can also be rephrased as $\forall b \prec hh(b \in a \to \textsc{Lev}(\downarrow b))$. Moreover, by $a \prec hh$ we know that $a =\uparrow (\P\P(a \cap hh))$, thus if we show that $(a \cap hh)$ is a history, $\downarrow a$ will be a level, contradiction. Assume a $b \prec (a \cap hh)$, that is, $b \in a \land b \prec hh$. Since $b \in a$, $\downarrow b$ is a level and, since $b \prec hh$, $b =\uparrow (\P\P(b \cap hh))$. Fix some $x \in b$, that is, $x \prec\downarrow b$, but levels are transitive so $x \sqsubseteq\downarrow b$, hence $x \subseteq b$. Then, since $b \in a$, i.e., $b \prec\downarrow a$, we have $x \blacktriangleright\downarrow a$ as $\downarrow a$ is super-transitive ($a \prec hh$ + Corollary A.10 + Lemma A.3), and so $x \in a$. Therefore $b \subseteq a$, so $b =\uparrow (\P\P(b \cap hh)) =\uparrow (\P\P(b \cap (a \cap hh)))$, so $(a \cap hh)$ is a history. □

Lemma A.11. *If ss is a level, then $ss \approx \P\P(rr)$, with $rr \approx \|t : t \prec ss \land \textsc{Lev}(\downarrow t)\|$.*

Proof. ($\Leftarrow$). Assume ss is a level and $a \prec \P\P(rr)$, that is, $a \blacktriangleright rr$. So, from Def. 2.3 and the definition of rr we know that there is some $t \supseteq a$ with $t \prec rr$, that is, $t \prec ss$. But then $a \blacktriangleright ss$ and since levels are super-transitive $a \prec ss$. ($\Rightarrow$). Assume $a \prec ss$. By Def. 4.5 we know that there is a history hh such that $ss \approx \P\P(hh)$ and so $a \prec \P\P(hh)$. This means that $a \blacktriangleright hh$, that is, there is a $t \prec hh$ with $t \supseteq a$. Remember that by Fact A.4 $hh \preccurlyeq \P\P(hh)$ and so $hh \preccurlyeq ss$, which allows the use of Lemma A.9 to say that $\downarrow t$ is a level. This plus $t \prec hh \preccurlyeq ss$ yield $t \prec rr$. □

Lemma A.12. *Levels are $\ltimes$-comparable:*

$$\forall ss \forall tt((\textsc{Lev}(ss) \land \textsc{Lev}(tt)) \to (\uparrow ss \prec tt \lor ss \approx tt \lor \uparrow tt \prec ss))$$

Proof. Suppose, for reductio, that some levels are $\ltimes$-incomparable. Since Lemma A.8 applies trivially to levels, we can assume a $\ltimes$-minimal level ss which is $\ltimes$-incomparable with some level. This means that for any $r \prec ss$ with $\textsc{Lev}(\downarrow r)$, $\downarrow r$ is $\ltimes$-comparable with all levels. Another round of Lemma A.8 provides us with another $\ltimes$-minimal level tt which is $\ltimes$-incomparable with ss. I shall show that $ss \approx tt$, contradiction. Fix some $a \prec ss$. By Lemma A.11 there is some $r \prec ss$ with $\textsc{Lev}(\downarrow r)$ and $a \subseteq r$. Since $\downarrow r$ is $\ltimes$-comparable with all levels this is also true for tt. But, if $\downarrow r \approx tt$, then $r = \uparrow tt$ and so $\uparrow tt \prec ss$, contradicting choice of tt. On the other hand, if $\uparrow tt \prec \downarrow r$, since levels are transitive we'd have $\uparrow tt \prec ss$, contradicting again choice of tt. So it must be that $r \prec tt$. Since levels are super-transitive we have $a \prec tt$ and thus $ss \preccurlyeq tt$. A similar reasoning yields $tt \preccurlyeq ss$, hence $ss \approx tt$. □

Overall, Lemma A.8 and A.12 tell us that $\ltimes$ acts as a sort of "trans-type well-ordering".

Lemma A.13. *If some plural level is Φ, then there is a $\preccurlyeq$-minimal level which is Φ. Formally:* $\exists rr(\textsc{Lev}(rr) \wedge \Phi(rr)) \rightarrow \exists ss((\textsc{Lev}(ss) \wedge \Phi(ss)) \wedge \forall rr((\textsc{Lev}(rr) \wedge \Phi(rr)) \rightarrow (rr \preccurlyeq ss \rightarrow rr \approx ss)))$.

Proof. As Lemma A.8 trivially applies to levels, repeat its steps to obtain a $\ltimes$-minimal plural level ss. Assume there is a plural level tt such that $\Phi(tt)$ and $tt \preccurlyeq ss$. Since $\uparrow tt \not\prec ss$, by Lemma A.12 either $ss \approx tt$, in which case we are done, or $\uparrow ss \prec tt$. But levels are transitive so $\uparrow ss \sqsubseteq tt$, hence $ss \preccurlyeq tt$. □

In general, this lemma tells us that $\ltimes$-minimality implies $\preccurlyeq$-minimality.[67] While the former is, in a sense, more fundamental, we still need the latter to properly state the usual "intra-type" kind of well-ordering.

Lemma A.14. *All levels are comparable:*

$$\forall ss \forall tt((\textsc{Lev}(ss) \wedge \textsc{Lev}(tt)) \rightarrow (ss \preccurlyeq tt \vee ss \approx tt \vee tt \preccurlyeq ss)).$$

Proof. Follow the steps of Lemma A.12 substituting $\ltimes$-minimal with $\preccurlyeq$-minimal. □

We are now ready to state the fundamental theorem of Plural Level Theory:

Theorem A.15. *The levels are well-ordered by plural inclusion ($\preccurlyeq$).*

Proof. The theorem follows from Lemma A.13 together with Lemma A.14; □

As for LT, Lemmas A.8 and A.12, together with P-STRAT, allow us to consider the level at which the elements of a set first appear.

Definition A.16. If $\uparrow aa$ exists, ℓaa is the $\ltimes$-least level that contains aa: $aa \preccurlyeq \ell aa$ and $\forall ss((\textsc{Lev}(ss) \wedge aa \preccurlyeq ss) \rightarrow \uparrow ss \not\prec \ell aa)$.

[67] For reasons of type bookkeeping, namely the fact that we are moving between higher and lower types, the other direction seems to not obviously follow. While it is not an issue for the overall theory, I leave the question open for further investigations since it may provide interesting insights on these type raising/lowering phenomena.

This is a rather powerful tool since, together with the above results, it sanctions $\ltimes$-induction, which ultimately yields to the familiar $\in$-induction. This is also how we prove that no set is among the plurality from which it collapses, a feature that Oliver and Smiley missed.[68] Moreover, it also yields some intuitive properties of the levels, analogous to Button (2021, Lemma 3.12): e.g., $\uparrow aa \not\prec \ell aa$ parallels LT's 3.12(2) and expresses the "priority" of the elements of a set to the set itself.[69]

1.2. Bounded Pluralities

Proposition 5.5, PLT is enough to sanction the Kuratowski notation for ordered pairs. We can then use P-COMP to define "relational pluralities" as pluralities of ordered pairs. In fact, we can do more and define "functional pluralities" as relational pluralities where no two ordered pairs share the first element. Finally, we can relativize this notion to a given set to define the plurality of functions on that set:

Definition A.17 (FUNCTIONAL PLURALITY)**.** Given a set a, we define a FUNCTIONAL PLURALITY ON a as

$$ff^a :\approx \|x : \forall y \in a(\exists z(x = \langle y, z\rangle) \land \forall x, x', z, z'((x = \langle y, z\rangle \land x' = \langle y, z'\rangle) \to z = z'))\|$$

We can now define the "plural image" of a functional plurality:

Definition A.18 (PLURAL IMAGE)**.** Given a set a and the functional plurality ff^a on a, the PLURAL IMAGE OF ff^a is $ff^{[a]} :\approx \|y : (\exists x \in a)(\exists p \prec ff^a)p = \langle x, y\rangle\|$.

Remember that all these pluralities exist simply by P-COMP and since in $\mathcal{L}_{\prec,\ltimes}$ we are availing ourselves of unrestricted plural quantification we can freely quantify over them to obtain the final principle.

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[68] I leave this as an exercise to the reader.
[69] I redirect the reader to Button's paper for the proofs of these features since they match almost perfectly and are not so interesting.

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