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The Spinning Electron

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Abstract. The notion introduced by Ohanian that *spin is a wave property* is implemented, both in Dirac and in Schrödinger quantum mechanics. We find that half-integer spin is the consequence of azimuthal dependence in two of the four spinor components, relativistically and non-relativistically. In both cases the spinor components are free particle wavepackets; the total wavefunction is an eigenstate of the total angular momentum in the direction of net particle motion. In the non-relativistic case we make use of the Lévy-Leblond result that four coupled non-relativistic wave equations, equivalent to the Pauli-Schrödinger equation, represent particles of half-integer spin, with g-factor 2. An example of an exact Gaussian solution of the non-relativistic equations is illustrated.

Keywords: electron, spin, spinor.

1. INTRODUCTION

In his article "What is spin" [1], Ohanian argues that 'spin may be regarded as an angular momentum generated by a circulating flow of energy in the wave field of the electron', and that 'the spin of the electron has a close classical analog: It is an angular momentum of exactly the same kind as carried by the wave field of a circularly polarized electromagnetic wave.' Ohanian credits Belifante [2] for establishing that 'this picture of spin is valid not only for electrons but also for photons, vector mesons, and gravitons.'

Dirac [3,4] regarded his four-by four matrices as 'new dynamical variables…describing some internal motion of the electron, which for most purposes may be taken to be the spin of the electron postulated in previous theories' [4]. This is how the concept of spin is presented in most texts, as intrinsically relativistic, a mysterious internal angular momentum for which there is no classical analogue. For example, in his "Introduction to quantum mechanics" [5] Griffiths states '…the electron also carries *another* form of angular momentum, which has nothing to do with motion in space (and which is not, therefore, described by any function of the position variables r,θ,ϕ) but which is somewhat analogous to classical spin…'.

We shall construct, for a general *relativistic or non-relativistic* wavepacket, an eigenstate of the component of total angular momentum in the net direction of propagation, with eigenvalue $\hbar/2$. Such eigenstates are four-component spinors, of which two components have $e^{i\phi}$ azimuthal dependence. In these formulations the phenomenon of spin is incorporated into ordinary space-time: the twist is in the azimuthal dependence of two of the wavefunctions. To the question: *what does a spinning electron look like*? we answer, in brief, that spin in the spinor formulation, relativistic or nonrelativistic, resides in the azimuthal dependence of two of the spinor components. This contrasts with the usual spin-space formulation, and the decoupling of spin from space-time.

In Sections 2 we construct general relativistic wavepackets with spin half; these are four-component spinors. An important aspect of spin is that *it is not purely a relativistic effect*: Levy-Léblond [6] has proved that the Galileo group has irreducible representations with non-zero spin. A Reviewer has pointed out that Galindo and del Rio [7] show that Galilean fermions are possible, with a four-component spinor linearization of the non-relativistic wave equation and a correct (to lowest order) g-factor. The Galindo and del Rio paper anticipates some of the work of Lévy-Leblond [6] and Gould [14].

Levy-Léblond's four-component nonrelativistic spinors are implemented in Section 3, to construct general angular momentum eigenstates with spin half. An explicit example of a non-relativistic spinning wavepacket is illustrated in Section 4.

2. DIRAC SPINORS

The wavefunction $\Psi(\boldsymbol{r},t)$ of an electron wavepacket in free space is to satisfy the Dirac equation

$$H\Psi(\boldsymbol{r},t)=i\hbar\partial_t\Psi(\boldsymbol{r},t),\quad H=c\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta mc^2,\quad \boldsymbol{p}=-i\hbar\nabla \tag{2.1}$$

The 4×4 matrices $\boldsymbol{\alpha},\beta$ are written in terms of the Pauli spin matrices $\sigma_x,\sigma_y,\sigma_z$ and the unit 2×2 matrix I as

$$\boldsymbol{\alpha}=\begin{pmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix},\ \sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix} \tag{2.2}$$

The wave equation (2.1) thus consists of four coupled first-order partial differential equations.

We consider wavepacket motion, predominantly along the z direction. In cylindrical polar coordinates $\rho=(x^2+y^2)^{\frac{1}{2}}$ is the distance from the z-axis, ϕ is the azimuthal angle, and

$$\frac{i}{\hbar}(p_x\pm ip_y)=\partial_x\pm i\partial_y=e^{\pm i\phi}(\partial_\rho\pm i\rho^{-1}\partial_\phi) \tag{2.3}$$

The four time-dependent free-space equations for the spinor Ψ read, with $mc/\hbar=K$,

$$(\partial_{ct}+iK)\psi_1+e^{-i\phi}(\partial_\rho-i\rho^{-1}\partial_\phi)\psi_4+\partial_z\psi_3=0 \tag{2.4a}$$

$$(\partial_{ct}+iK)\psi_2+e^{i\phi}(\partial_\rho+i\rho^{-1}\partial_\phi)\psi_3-\partial_z\psi_4=0 \tag{2.4b}$$

$$(\partial_{ct}-iK)\psi_3+e^{-i\phi}(\partial_\rho-i\rho^{-1}\partial_\phi)\psi_2+\partial_z\psi_1=0 \tag{2.4c}$$

$$(\partial_{ct}-iK)\psi_4+e^{i\phi}(\partial_\rho+i\rho^{-1}\partial_\phi)\psi_1-\partial_z\psi_2=0 \tag{2.4d}$$

When the spinor components ψ_j are independent of ϕ, solutions exist only for the ψ_j also independent of ρ. These are the well-known plane wave solutions $\psi_j=a_je^{i(qz-\omega t)}$, where the wavenumber q and the energy $\hbar\omega$ are constrained by $(\omega/c)^2=K^2+q^2$. To attain localized wavepacket solutions, we need to consider azimuthal dependence.

The angular momentum operator $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$ does not commute with the Hamiltonian, but the combination $\boldsymbol{J}=\boldsymbol{L}+\frac{\hbar}{2}\boldsymbol{\Sigma}$ does, where $\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma}&0\\0&\boldsymbol{\sigma}\end{pmatrix}$. The z component of the total angular momentum operator is

$$J_z=L_z+\frac{\hbar}{2}\begin{pmatrix}\sigma_z&0\\0&\sigma_z\end{pmatrix}=-i\hbar\,diag(1,1,1,1)\partial_\phi+ +\frac{\hbar}{2}diag(1,-1,1,-1) \tag{2.5}$$

Let the spinor components ψ_j have azimuthal dependence $e^{i\nu_j\phi}$; the J_z eigenstate equations for ψ_1,ψ_2 read

$$\begin{pmatrix}-i\partial_\phi+1/2&0\\0&-i\partial_\phi-1/2\end{pmatrix}\begin{pmatrix}e^{i\nu_1\phi}\\e^{i\nu_2\phi}\end{pmatrix}= =\begin{pmatrix}(\nu_1+1/2)e^{i\nu_1\phi}\\(\nu_2-1/2)e^{i\nu_2\phi}\end{pmatrix} \tag{2.6}$$

This will be an eigenstate of J_z if $\nu_1+1/2=\nu_2-1/2$, $\nu_2-\nu_1=1$, with eigenvalue $(\nu_1+1/2)\hbar$. Similarly for ψ_3,ψ_4 we shall have an eigenstate of J_z if $\nu_3+1/2=\nu_4-1/2$, $\nu_4-\nu_3=1$, with eigenvalue $(\nu_3+1/2)\hbar$. Hence the choice $\nu_{1,3}=0$, $\nu_{2,4}=1$ makes Ψ an eigenstate of J_z with eigenvalue $\hbar/2$. (The choice $\nu_{1,3}=-1$, $\nu_{2,4}=0$ makes Ψ an eigenstate of J_z with eigenvalue $-\hbar/2$.) It is necessary to have integer ν_j, since the spinor components are in real space-time (not in some abstract spin space) so we must have $\psi_j(\phi+2\pi)=\psi_j(\phi)$. The eigenvalues of J_z are thus $\pm\hbar/2,\pm3\hbar/2$ etc.

With spinor components $\psi_{1,3}=f_{1,3}(\rho,z,t),\psi_{2,4}=e^{i\phi}f_{2,4}(\rho,z,t)$, the azimuthal dependence cancels out, and the equations (2.4) read

$$(\partial_{ct}+iK)f_1+(\partial_\rho+\rho^{-1})f_4+\partial_zf_3=0 \tag{2.7a}$$

$$(\partial_{ct}+iK)f_2+\partial_\rho f_3-\partial_z f_4=0 \quad (2.7b)$$

$$(\partial_{ct}-iK)f_3+(\partial_\rho+\rho^{-1})f_2+\partial_z f_1=0 \quad (2.7c)$$

$$(\partial_{ct}-iK)f_4+\partial_\rho f_1-\partial_z f_2=0 \quad (2.7d)$$

The combination $(\partial_{ct}-iK)$(2.7a)-$(\partial_\rho+\rho^{-1})$(2.7d)-$\partial_z$(2.7c) gives

$$(\partial^2_{ct}+K^2-\partial^2_\rho-\rho^{-1}\partial_\rho-\partial^2_z)f_1(\rho,z,t)=0 \quad (2.8)$$

Likewise $(\partial_{ct}-iK)$(2.7b)-∂_ρ(2.7c)+∂_z(2.7d) gives us

$$(\partial^2_{ct}+K^2-\partial^2_\rho-\rho^{-1}\partial_\rho+\rho^{-2}-\partial^2_z)f_2(\rho,z,t)=0 \quad (2.9)$$

The equations (2.8) and (2.9) are solved respectively by

$$e^{i(qz-\omega t)}J_0(k\rho),\quad e^{i(qz-\omega t)}J_1(k\rho),\quad k^2+q^2+K^2=(\omega/c)^2 \quad (2.10)$$

The function f_3 satisfies the same equation as f_1, and f_4 satisfies the same equation as f_2. The transverse and longitudinal wavenumbers k and q are real, and $\omega\geq cK$, or $\hbar\omega\geq mc^2$. The wavenumbers $k\geq0$ and $q\geq0$ are related to $K=mc/\hbar$ and ω by $k^2+q^2+K^2=(\omega/c)^2$; the maximum value of both k and q is $Q=\sqrt{(\omega/c)^2-K^2}$. Hence the general form of the spinor eigenstate of J_z with eigenvalue $\hbar/2$ is

0

$$\psi_{1,3}(\rho,z,t)=\int_{cK}^\infty d\omega\int_0^Q dk\, A_{1,3}(\omega,k)e^{i(qz-\omega t)}J_0(k\rho) \quad (2.11)$$

$$\psi_{2,4}(\rho,\phi,z,t)=e^{i\phi}\int_{cK}^\infty d\omega\int_0^Q dk\, A_{2,4}(\omega,k)e^{i(qz-\omega t)}J_1(k\rho) \quad (2.12)$$

These are analogues of the acoustic and electromagnetic wavepackets, for which simple closed forms exist ([8], Section 2.6). The author has not found amplitudes $A_j(\omega,k)$ which lead to closed forms for the relativistic spinor components. Bessel beam wavefunctions (not localized enough transversely to have finite energy per unit length) have been studied by Bliokh et al. [9].

3. NON-RELATIVISTIC SPINORS

Lévy-Leblond [6] has shown that four coupled non-relativistic wave equations, equivalent to the Schrödinger equation, are spinors representing spin 1/2 particles, with g-factor 2 (see also Greiner [10]). We shall again construct a general eigenstate of J_z with eigenvalue $\hbar/2$: it is a four-component spinor. It is based on localized wavepacket solutions of the time-dependent Schrödinger equation, with no restriction on the wavepacket parameters. In Section 4 we shall explore some properties of exact Gaussian solutions of the equations satisfied by the spinor components.

Let $\Psi(\boldsymbol{r},t)$ be the four-component spinor, $\Psi=\begin{pmatrix}\psi\\ \chi\end{pmatrix}$, with ψ,χ each having two components. The Lévy-Leblond non-relativistic coupled spinor equations are, with $E=i\hbar\partial_t$, $\boldsymbol{p}=-i\hbar\nabla$,

$$E\psi+\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi=0,\quad \boldsymbol{\sigma}\cdot\boldsymbol{p}\psi+2m\chi=0 \quad (3.1)$$

$\boldsymbol{\sigma}$ are, as before, the Pauli spin matrices defined in (2.2).

Note that the ψ,χ in (3.1) have dimension differing by a speed; we could make them the same by inserting factors $e^2/\hbar$ or c in front of χ, but choose not to do so, in order keep the Lévy-Leblond formulation. Note also that the lower spinor component χ can be eliminated, giving the Pauli-Schrödinger equation $E\psi=\frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2\psi$, with Hamiltonian $H=\frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2$.

For comparison, the Dirac equations (2.1), with $\psi_u=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, $\psi_v=\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$, may be written in the form

$$(E-mc^2)\psi_u=c\boldsymbol{\sigma}\cdot\boldsymbol{p}\psi_v,\quad c\boldsymbol{\sigma}\cdot\boldsymbol{p}\psi_u=(E+mc^2)\psi_v \quad (3.2)$$

The non-relativistic limit is obtained from (3.2) by setting $\psi_j(\boldsymbol{r},t)=e^{-imc^2t/\hbar)}F_j(\boldsymbol{r},t)$. Then $E\psi_j=i\hbar\partial_t\ \psi_j=e^{-\frac{imc^2t}{\hbar}}(mc^2+i\hbar\partial_t)F_j$, and the equations (3.2) have the dominant terms

$$EF_u=c\boldsymbol{\sigma}\cdot\boldsymbol{p}F_v,\quad c\boldsymbol{\sigma}\cdot\boldsymbol{p}F_u=2mc^2F_v \quad (3.3)$$

These are the same as (3.1) if we identify F_u with ψ, and cF_u with $-\chi$.

Returning to solutions of the Lévy-Leblond equations (3.10), we write $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, $\chi=\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$, and consider wavepacket motion, predominantly along the direction, but of course converging onto or diverging from the focal region, which we shall centre at the space-time origin. Again in cylindrical polar coordinates ρ,ϕ, and with use of (2.3), the four time-dependent free-space equations (3.1) for the spinor Ψ read

$$-\partial_t\psi_1+e^{-i\phi}(\partial_\rho-i\rho^{-1}\partial_\phi)\psi_4+\partial_z\psi_3=0 \quad (3.4a)$$

$$-\partial_t\psi_2+e^{i\phi}(\partial_\rho-i\rho^{-1}\partial_\phi)\psi_3+\partial_z\psi_4=0 \quad (3.4b)$$

$$\frac{2im}{\hbar}\psi_3+e^{-i\phi}(\partial_\rho-i\rho^{-1}\partial_\phi)\psi_2+\partial_z\psi_1=0 \quad (3.4c)$$

$$\frac{2im}{\hbar}\psi_4+e^{i\phi}(\partial_\rho-i\rho^{-1}\partial_\phi)\psi_1+\partial_z\psi_2=0 \quad (3.4d)$$

When the spinor components ψ_j are independent of ϕ, solutions exist only for the ψ_j also independent of ρ. These are the plane wave solutions $\psi_j=a_je^{i(qz-\omega t)}$, where the wavenumber k and the energy $\hbar\omega$ are constrained by

$\hbar\omega=\hbar^2q^2/2m$. To attain localized wavepacket solutions, we need to consider azimuthal dependence.

The angular momentum operator $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$ does not commute with the free-particle Hamiltonian $H=\frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2$, but the combination $\boldsymbol{J}=\boldsymbol{L}+\frac{\hbar}{2}\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}$ does, as may be verified from the commutators $\boldsymbol{\sigma}\times\boldsymbol{\sigma}=2i\boldsymbol{\sigma}$, $[L,\boldsymbol{\sigma}\cdot\boldsymbol{p}]=i\hbar\boldsymbol{\sigma}\times\boldsymbol{p}$, $[\boldsymbol{\sigma},\boldsymbol{\sigma}\cdot\boldsymbol{p}]=-2i\boldsymbol{\sigma}\times\boldsymbol{p}$. $\boldsymbol{J}$ satisfies the angular momentum commutation relations $\boldsymbol{J}\times\boldsymbol{J}=i\hbar\boldsymbol{J}$. The z component of the total angular momentum operator is again

$$J_z=L_z+\frac{\hbar}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix}=-i\hbar\, diag(1,1,1,1)\partial_\phi+\frac{\hbar}{2}\, diag(1,-1,1,-1) \quad (3.5)$$

We shall now construct the non-relativistic spinor eigenstates of J_z.

Let the spinor components ψ_j have azimuthal dependence $e^{i\nu_j\phi}$; the J_z eigenstate equations for ψ_1,ψ_2 are the same as in (2.6):

$$\begin{pmatrix}-i\partial_\phi+1/2 & 0\\ 0 & -i\partial_\phi-1/2\end{pmatrix}\begin{pmatrix}e^{i\nu_1\phi}\\ e^{i\nu_2\phi}\end{pmatrix}=\begin{pmatrix}(\nu_1+1/2)e^{i\nu_1\phi}\\ (\nu_2-1/2)e^{i\nu_2\phi}\end{pmatrix} \quad (3.6)$$

The equations (3.5) and (3.6) have the same form as in the relativistic case, equations (2.5) and (2.6). Hence as before the choice $\nu_{1,3}=0$, $\nu_{2,4}=1$ makes Ψ an eigenstate of J_z with eigenvalue $\hbar/2$ and the choice $\nu_{1,3}=-1$, $\nu_{2,4}=0$ makes Ψ an eigenstate of J_z with eigenvalue $-\hbar/2$. With spinor components $\psi_{1,3}=f_{1,3}(\rho,z,t)$, $\psi_{2,4}=e^{i\phi}f_{2,4}(\rho,z,t)$, the equations (3.4) read

$$-\partial_t f_1+(\partial_\rho+\rho^{-1})f_4+\partial_z f_3=0 \quad (3.7a)$$

$$-\partial_t f_2+\partial_\rho f_3-\partial_z f_4=0 \quad (3.7b)$$

$$\frac{2im}{\hbar}f_3+(\partial_\rho+\rho^{-1})f_2+\partial_z f_1=0 \quad (3.7c)$$

$$\frac{2im}{\hbar}f_4+\partial_\rho f_1-\partial_z f_2=0 \quad (3.7d)$$

The last two equations give $f_{3,4}$ in terms of derivatives of $f_{1,2}$, which in turn satisfy the free-space Schrödinger equation for azimuthal orbital quantum number 0 and 1:

$$(i\hbar\partial_t+\frac{\hbar^2}{2m}[\partial_\rho^2+\rho^{-1}\partial_\rho+\partial_z^2])f_1(\rho,z,t)=0 \quad (3.8)$$

$$(i\hbar\partial_t+\frac{\hbar^2}{2m}[\partial_\rho^2+\rho^{-1}\partial_\rho-\rho^{-2}+\partial_z^2])f_2(\rho,z,t)=0 \quad (3.9)$$

Equations(3.8) and (3.9) are satisfied by $J_n(\kappa\rho)e^{in\phi}e^{iqz}e^{-i\hbar k^2t/2m}$, with n=0,1 respectively, and $\kappa^2+q^2=k^2$; J_n are the regular Bessel functions of order n. Hence spinor components of forward-propagating wavepackets have the form

$$e^{in\phi}\int_0^\infty dk\, e^{-i\hbar k^2t/2m}\int_0^k dq\, F_n(k,q)e^{iqz}J_n(\kappa\rho) \quad (\kappa^2+q^2=k^2) \quad (3.10)$$

The amplitudes $F_n(k,q)$ are complex functions, subject only to the existence of the norm and of the expectation values of energy and momentum of the wave packet. A similar expression gives the wavefunctions of scalar and of electromagnetic pulses [8].

To sum up this Section: a general non-relativistic eigenstate of J_z with eigenvalue $\hbar/2$ has been found: it is a four-component spinor, of which two components have 'twist', with $e^{i\phi}$ azimuthal dependence. In this formulation the spin resides in the azimuthal dependence of two of the wavefunctions, in real space-time.

Any spinor based on localized wavepacket solutions of the time-dependent Schrödinger equation, constructed as above, will be an eigenstate of J_z with eigenvalue $\hbar/2$. The next Section gives an explicit example. Stationary states (energy eigenstates) of the hydrogen atom are briefly discussed in Appendix A.

4. SPINNING GAUSSIAN WAVEPACKETS

A free-particle wavepacket solution of Schrödinger's time-dependent equation dates back to the early days of quantum mechanics (Kennard [11], Darwin [12]). This is the Gaussian wavepacket. It is a compact exact solution, but with a physical flaw, to be discussed below. For propagation along the axis, and with cylindrical symmetry, it has the form

$$g(\rho,z,t)=b^{3/2}[b+i\nu t]^{-3/2}exp\{iQ(z-\frac{ut}{2})-\frac{\rho^2+(z-ut)^2}{4b[b+i\nu t]}\} \quad (4.1)$$

The Gaussian wavepacket (4.1) is normalized so that $g^*g=1$ at the space-time origin. In (4.1) the spatial origin $\rho=0$, $z=0$ is the position of maximal $|g|$ a time t=0, Q is the dominant z component wavenumber, m is the mass of the particle, $u=\hbar Q/m$ is the group speed, and $\nu=\hbar/2mb$ is the spreading speed. The length b gives the spread of the wavepacket at t=0. Earlier and later the longitudinal and lateral spread of the packet is greater, proportional to $[b^2+(\nu t)^2]^{1/2}$. Thus $\rho=0$, $z=0$ can be thought of as the centre of the focal region of the wavepacket, occupied at t=0. As t increases towards zero the wavepacket converges to its most compact form, reaches it at t=0, and then expands as it continues to propagate in the positive z direction. The packet used by Ohanian [1] is equivalent to (4.1) evaluated at Q=0 (zero momentum expectation value) and t=0.

For the Gaussian wavepacket g the momentum operator has the expectation values (see for example [13])

$$\langle p_z\rangle=-i\hbar\langle\partial_z\rangle=\hbar Q,\quad \langle p_x\rangle=0=\langle p_y\rangle,\quad \langle p^2\rangle= =\langle-\hbar^2\nabla^2\rangle=\hbar^2\left(Q^2+\frac{3}{4b^2}\right) \tag{4.2}$$

The wavepacket g is neither an energy nor a momentum eigenstate, but it is an eigenstate of the orbital angular momentum operator $L_z=xp_y-yp_x=-i\hbar(x\partial_y-y\partial_x)=-i\hbar\partial_\phi$. The orbital angular momentum eigenvalue is zero, because g is independent of the azimuthal angle ϕ. Eigenstates of the z component of orbital angular momentum, with eigenvalues which are integer multiples of ħ, may be generated from any such g by differentiation, as shown in [13].

The probability density of the scalar wavepacket is g^*g: the probability that the particle described by $g(\boldsymbol{r},t)$ is within the volume element d^3r is $d^3r\ g^*g$. The *norm* $N=\int d^3r\ g^*g$ (integration over all of space) is independent of time. The probability density flux, or the probability current density vector $\boldsymbol{S}$, satisfies the conservation law

$$\nabla.\boldsymbol{S}+\partial_t(g^*g)=0,\quad \boldsymbol{S}(\boldsymbol{r},t)=\frac{\hbar}{m}Im(g^*\nabla g) \tag{4.3}$$

What are the corresponding relations for spinors? The conservation law is now (Lévy-Leblond [6], Section IIIe, and Appendix B)

$$\nabla.\boldsymbol{S}+\partial_t(\psi^+\psi)=0 \tag{4.4}$$

$$\boldsymbol{S}(\boldsymbol{r},t)=-\psi^+\boldsymbol{\sigma}\chi-\chi^+\boldsymbol{\sigma}\psi=\frac{\hbar}{m}Im(\psi^+\nabla\psi)+\frac{\hbar}{2m}\nabla\times(\psi^+\boldsymbol{\sigma}\psi) \tag{4.5}$$

The first term in the second expression for $\boldsymbol{S}$ corresponds to the Schrödinger current in (4.3), the second is a spin current. Ohanian [1] derived the relativistic analogue of last term in (4.5). He showed that it leads, in the nonrelativistic limit, to an azimuthal current. In his words, "such a circulating flow of energy will give rise to an angular momentum. This angular momentum is the spin of the electron."

We shall calculate the radial, azimuthal, and longitudinal components of the probability current density, S_ρ, S_ϕ, S_z in the simplest case, in which the spinor components are $\psi_1=f_1(\rho,z,t)$, $\psi_2=0$, $\psi_3\sim\partial_z\psi_1$, $\psi_4\sim e^{i\phi}\partial_\rho\psi_1$. From Appendix B, the components of the probability current density are given by

$$\frac{m}{\hbar}S_\rho=Im\{f^*_1\partial_\rho f_1\},\quad \frac{m}{\hbar}S_\phi=-\frac{1}{2}\partial_\rho|f_1|^2,\quad \frac{m}{\hbar}S_z=Im\{f^*_1\partial_z f_1\} \tag{4.6}$$

With $f_1(\rho,z,t)=g(\rho,z,t)$ the probability density and current components are given by

$$g^*g=b^3[b^2+(vt)^2]\ exp\left\{-\frac{\rho^2+(z-ut)^2}{2[b^2+(vt)^2]}\right\} \tag{4.7}$$

$$S_\rho=\frac{\rho t\left(\frac{\hbar}{2mb}\right)^2}{b^2+(vt)^2}g^*g,\quad S_\phi=\frac{\frac{\hbar\rho}{4m}}{b^2+(vt)^2}g^*g,\quad S_z=\frac{\frac{\hbar b^2}{m}\left(Q+\frac{vtz}{2b^3}\right)}{b^2+(vt)^2}g^*g \tag{4.8}$$

The components S_ρ, S_z are the same for the scalar wavepacket, the azimuthal component S_ϕ is zero in the scalar case based on g. The conservation law (4.4) is satisfied.

A problem with the Gaussian solution is apparent in S_z: for positive z and negative t (or vice versa) the longitudinal component is negative if the magnitude of vtz exceeds that of $2Qb^3$. The probability current then flows backward. Far from the focal region (here centred on the space-time origin) there should be no backward flow for free-space propagation. Note that the Gaussian wavepacket cannot be put in the purely forward-propagating form (3.10).

Nevertheless, the Gaussian packets demonstrate the azimuthal current component which arises in the spinor formulation. Figures 1 and 2 show the current components in the focal plane, and at a transverse plane cutting through the wavepacket center at a later time. The azimuthal part gives the electron wavepacket its spin.

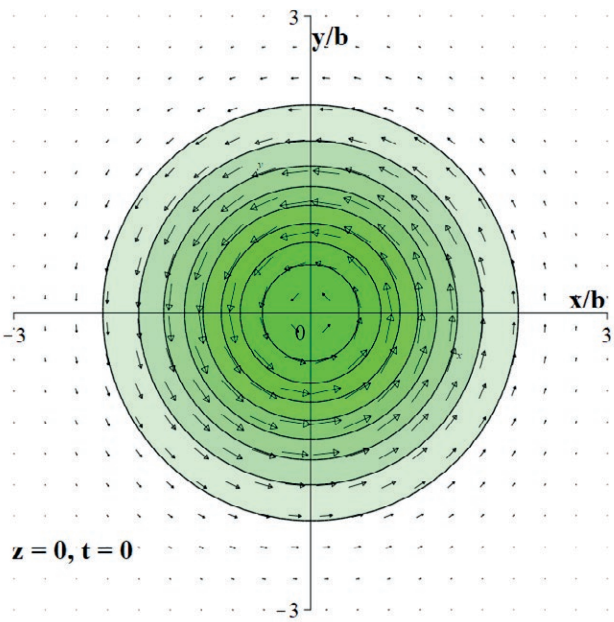

Figure 1. Focal plane section through a Gaussian spinor wavepacket, at t=0. The contours give the probability density, the arrows the transverse current density (the longitudinal current is not shown). The direction of motion is out of the page. The transverse current density is purely azimuthal at this instant.

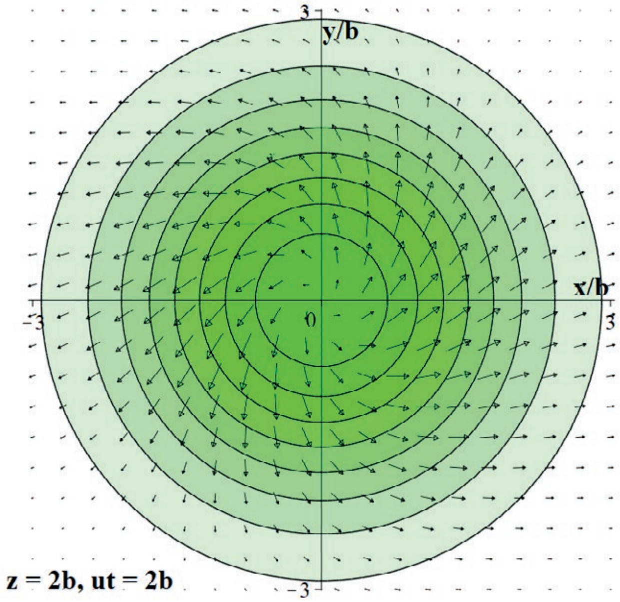

Figure 2. Gaussian spinor wavepacket, at $z=ut=2b$. The transverse current density now has radial and azimuthal components. The group speed is u, so the section is through the centre of the wavepacket. The longitudinal current density is not shown.

5. SUMMARY

The spinning electron may be described by a four-component spinor, depending on space and time coordinates, in both relativistic and non-relativistic quantum theory. The non-relativistic quantum theory and its azimuthal dependence is similar to the relativistic Dirac spinor formulation of Section 2. In both cases the spin is contained in the azimuthal dependence of wavefunctions in ordinary space-time. Gould [14] used the Hamiltonian $H=\frac{1}{2m}(\boldsymbol{\sigma}.\boldsymbol{p})^2$ to show that the magnetic moment follows (correct to lowest order), just as in the Lévy-Leblond spinor formulation. There is thus an alternative formulation to the usual 'spin degree of freedom', and the total wavefunction being a product of space and spin parts, as is done in nonrelativistic quantum theory. Nevertheless, the non-relativistic decoupling of space and spin is usually simpler, as is illustrated by the spinor version of the Hydrogen atom, Appendix A.

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APPENDIX A. THE HYDROGEN ATOM IN SPINOR FORM

The equations (3.1) become, with E now an energy eigenvalue, no longer a time derivative,

$$(E+\frac{e^2}{r})\psi+\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi=0, \quad \boldsymbol{\sigma}\cdot\boldsymbol{p}\psi+2m\chi=0 \tag{A.1}$$

$$\left[\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2}{2m}-\frac{e^2}{r}\right]\psi=E\psi \quad \text{or} \quad [\frac{-\hbar^2}{2m}\nabla^2-e\frac{e^2}{r}]\psi=E\psi \tag{A.2}$$

Considering the non-degenerate ground state, with J_z eigenvalue $\frac{\hbar}{2}$, ψ_1 and ψ_2 must satisfy the same equation. This is not possible if we choose ψ_2 to have azimuthal dependence $e^{i\phi}$, as in Section 3, unless we also take ψ_2 to be zero. The ground state spinor now consists of ψ_1, the hydrogenic ground state 1S, and $\psi_2=0$, $\psi_3\sim\partial_z\psi_1$, $\psi_4\sim e^{i\phi}\partial_\rho\psi_1$. Because the Lévy-Leblond probability density is defined in terms of the first two spinor components ψ_1,ψ_2, and the probability density current can be expressed in terms of ψ_1,ψ_2, the hydrogenic ground state is, at least in the probability density and the probability density current, equivalent to the scalar ground state. The azimuthal dependence is hidden in the fourth spinor component.

For the first excited states we have a choice of 2S and 2P. The former is set up as above, the latter with $\psi_1=0$, and ψ_2 with $e^{\pm i\phi}$ dependence. Lévy-Leblond [15] and Mita [16] discuss the electron probability current of the 'stationary' states.

APPENDIX B. PROBABILITY DENSITY AND FLUX

In the Dirac case (Section 2), $\Psi^+\Psi$ is the probability density, and $\boldsymbol{S}=c\Psi^+\boldsymbol{\alpha}\Psi$, with $\boldsymbol{\alpha}$ is defined in (2.2). In the nonrelativistic formulation of Lévy-Leblond we have a time derivative of ψ but not of χ: $i\hbar\partial_t\psi+\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi=0$, $\boldsymbol{\sigma}\cdot\boldsymbol{p}\psi+2m\chi=0$, or $\partial_t\psi-\boldsymbol{\sigma}\cdot\nabla\chi=0,-i\hbar\boldsymbol{\sigma}\cdot\nabla\psi+2m\chi=0$. To keep the norm time-independent Lévy-Leblond defines the probability density in terms of ψ only, as $\psi^+\psi$. The conservation law is now (Lévy-Leblond [6], Section IIIe)

$$\nabla.\boldsymbol{S}+\partial_t(\psi^+\psi)=0 \tag{B.1}$$

$$\partial_t(\psi^+\psi)=\psi^+(\boldsymbol{\sigma}\cdot\nabla\chi)+(\nabla\chi^+\cdot\boldsymbol{\sigma})\psi=\nabla\cdot(\psi^+\boldsymbol{\sigma}\chi+\chi^+\boldsymbol{\sigma}\psi) \tag{B.2}$$

Hence $\boldsymbol{S}(\boldsymbol{r},t)=-(\psi^+\boldsymbol{\sigma}\chi+\chi^+\boldsymbol{\sigma}\psi)$. We may express this current purely in terms of the top two spinor components ψ, since $\chi=\frac{i\hbar}{2m}\boldsymbol{\sigma}\cdot\nabla\psi$. This gives

$$\boldsymbol{S}(\boldsymbol{r},t)=\frac{\hbar}{2im}\{\psi^+\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\nabla\psi)-(\boldsymbol{\sigma}\cdot\nabla\psi)^+\boldsymbol{\sigma}\psi\} \tag{B.3}$$

On using the commutation relations of the Pauli matrices, $\boldsymbol{\sigma}\times\boldsymbol{\sigma}=2i\boldsymbol{\sigma}$, the probability density current becomes

$$\boldsymbol{S}(\boldsymbol{r},t)=\frac{\hbar}{2im}[\psi^+\nabla\psi-(\nabla\psi^+)\psi]+\frac{\hbar}{2m}\nabla\times(\psi^+\boldsymbol{\sigma}\psi) \tag{B.4}$$

The first term in this expression for $\boldsymbol{S}$ corresponds to the Schrödinger current in (3.3), the second is a spin current, which gives the correct g factor at leading order [6]. The spin term is the curl of a vector, and so does not contribute to the conservation law (B.1). See also Landau and Lifshitz [17] Section 114, and Mita [16] for the spin current term.

We shall calculate the radial, azimuthal, and longitudinal components of the probability current density, S_ρ,S_ϕ,S_z. The corresponding spin matrix components are

$$\sigma_\rho=\boldsymbol{\sigma}.\hat{\boldsymbol{\rho}}=\begin{pmatrix}0 & e^{-i\phi}\\ e^{i\phi} & 0\end{pmatrix}, \quad \sigma_\phi=\boldsymbol{\sigma}.\hat{\boldsymbol{\phi}}=\begin{pmatrix}0 & -ie^{-i\phi}\\ ie^{i\phi} & 0\end{pmatrix}, \quad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{B.5}$$

Let f_1,f_2 be solutions of (3.8) and (3.9), respectively, and $\psi_1=f_1$, $\psi_2=e^{i\phi}f_2$. We can set $f_2=a\partial_\rho f_1$ [12]; a is a length parameter. We shall first calculate $\psi^+\boldsymbol{\sigma}\psi$; this has the cylindrical components $(2a\mathrm{Re}\{f^{*1}\partial_\rho f_1\}$, $2a\mathrm{Im}\{(\partial_\rho f^*{}_1)f_1\}$, $|f_1|^2-a^2|\partial_\rho f_1|^2)$. Note that there is no ϕ dependence. The curl of this vector is

$$\nabla\times(\psi^+\boldsymbol{\sigma}\psi)=(-2a\partial_z\mathrm{Im}\{(\partial_\rho f_1^*)f_1\},\ 2a\partial_z\mathrm{Re}\{f_1^*\partial_\rho f_1\}-\partial_\rho[|f_1|^2-a^2|\partial_\rho f_1|^2],\ 2a\partial_\rho\mathrm{Im}\{(\partial_\rho f_1^*)f_1\}+2a\rho^{-1}\mathrm{Im}\{(\partial_\rho f_1^*)f_1\}) \tag{B.6}$$

When the length a is zero, just the azimuthal component remains, $\nabla\times(\psi^+\boldsymbol{\sigma}\psi)_{a=0}=(0,-\partial_\rho|f_1|^2,0)$. In that special case the Schrödinger current is proportional to $\mathrm{Im}\{f^*{}_1\nabla f_1\}=\mathrm{Im}\{f^*{}_1(\partial_\rho f_1,0,\partial_z f_1)\}$, and the components of the probability current density are given by

$$\frac{m}{\hbar}S_\rho=\mathrm{Im}\{f^*{}_1\partial_\rho f_1\}, \quad \frac{m}{\hbar}S_\phi=-\frac{1}{2}\partial_\rho|f_1|^2, \quad \frac{m}{\hbar}S_z=\mathrm{Im}\{f^*{}_1\partial_z f_1\} \tag{B.7}$$

As in the hydrogen ground state, the a=0 spinor now consists of ψ_1, and $\psi_2=0$, $\psi_3\sim\partial_z\psi_1$, $\psi_4\sim e^{i\phi}\partial_\rho\psi_1$. The fourth component contributes to the azimuthal current, and to the angular momentum.